Notes

Pre-Calculus
Sec. 5.2
Sum and Difference Formulas

The Sum and Difference Formula for Sine

•Same sign in a sine sandwich

Sum: $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Difference: $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

Condenses $\longleftrightarrow$ Expands

The Sum and Difference Formula for Cosine

Sum: $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Difference: $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

The Two Main Purposes for the Sum/Diff. Formulas:

1)Finding the **exact** values of trigonometric expressions with non common angles.

2)Simplifying expressions to obtain other identities.

non-calc

Ex. 1: Find the exact value of each of the following:

$a)\ \cos\underline{\underline{15}}^\circ$

$60^\circ - 45^\circ$

$45^\circ - 30^\circ$

$$= \cos(\overset{\alpha}{45^\circ} - \overset{\beta}{30^\circ})$$

expand

opp

$$= \cos\overset{\alpha}{45^\circ}\cos\overset{\beta}{30^\circ} + \sin\overset{\alpha}{45^\circ}\sin\overset{\beta}{30^\circ}$$

$$= \underbrace{\left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

Done another way

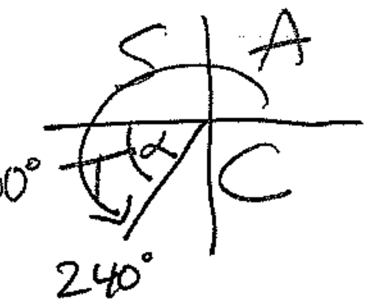
b) $\sin 285°$

$\underset{}{45°} + \underset{(60°)}{240°}$

$\underset{(45°)}{315°} - 30°$

$\underset{(45°)}{225°} + 60°$

$\alpha = 60°$



$= \sin(\overset{\alpha}{45°} + \overset{\beta}{240°})$

Expand w/rule

$= \sin\alpha \cos\beta \overset{\text{same}}{+} \cos\alpha \sin\beta$

$= \sin 45° \cos 240° + \cos 45° \sin 240°$

$= \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)}$

$-\frac{\sqrt{2}}{4} \underbrace{+ \left(-\frac{\sqrt{6}}{4}\right)}_{\text{combine signs}} = \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$

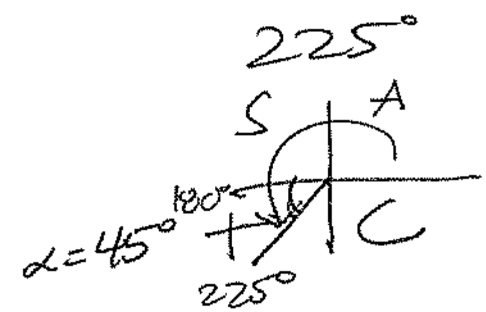
Done another way

b) $\sin 285°$

$45° + 240°$

$315° - 30°$

$225° + 60°$



$$= \sin(\overset{\alpha}{225°} + \overset{\beta}{60°})$$

expand

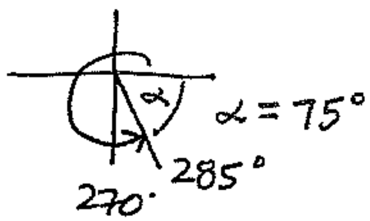
$$= \sin \overset{\alpha}{225°} \cos \overset{\beta}{60°} \overset{\text{same sign}}{+} \cos \overset{\alpha}{225°} \sin \overset{\beta}{60°}$$

$$= \underbrace{\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)} + \underbrace{\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= -\frac{\sqrt{2}}{4} + \left(-\frac{\sqrt{6}}{4}\right)$$

$$= \boxed{\frac{-\sqrt{2} - \sqrt{6}}{4}}$$

$$b)\ \sin 285^\circ = \overset{\text{Quadrant}}{-}\ \sin \overset{\text{reference angle}}{75^\circ}$$



S	A
T	C

$$= -\underset{\text{expand w/ rule}}{\left[\sin(30^\circ + 45^\circ)\right]}$$

$$= -\left[\sin 30^\circ \cos 45^\circ \overset{\text{same}}{+} \cos 30^\circ \sin 45^\circ\right]$$

$$= -\left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\left[\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right]$$

$$= \boxed{\frac{-\sqrt{2} - \sqrt{6}}{4}}$$

Ex.2: Simplify each of the following expressions.

Can you condense using a sum/difference formula?

$a)$ $\underbrace{\cos 50^\circ}_{\alpha} \underbrace{\cos 40^\circ}_{\beta} - \underbrace{\sin 50^\circ}_{\alpha} \underbrace{\sin 40^\circ}_{\beta}$ Pattern?

opp

Rule: $\cos(\alpha + \beta)$

$= \cos(50^\circ + 40^\circ)$

$= \cos 90^\circ$

$= \boxed{0}$

b) $\sin\overset{\alpha}{\frac{11\pi}{30}}\cos\overset{\beta}{\frac{\pi}{5}} - \sin\overset{\beta}{\frac{\pi}{5}}\cos\overset{\alpha}{\frac{11\pi}{30}}$

Pattern?

reorder

$\sin\overset{\alpha}{\frac{11\pi}{30}}\cos\overset{\beta}{\frac{\pi}{5}} - \cos\overset{\alpha}{\frac{11\pi}{30}}\sin\overset{\beta}{\frac{\pi}{5}}$

Sine Sandwich

same

$= \sin(\alpha - \beta)$

$= \sin\left(\frac{11\pi}{30} - \frac{\pi}{5}\cdot\frac{6}{6}\right)$

$= \sin\left(\frac{11\pi}{30} - \frac{6\pi}{30}\right)$

$= \sin\left(\frac{5\pi}{30}\right)$

$= \sin\left(\underset{30^\circ}{\frac{\pi}{6}}\right) = \boxed{\frac{1}{2}}$

The Sum and Difference Formula for Tangent

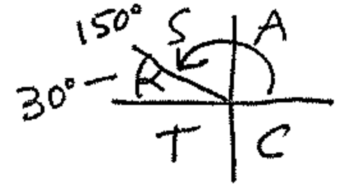
Sum: $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

Condense $\longleftrightarrow$ Expand

Difference: $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

Ex. 3: Find the exact value of each of the following:

$a)\ \tan 195^\circ$ $\qquad 150^\circ + 45^\circ$ $\underset{(30^\circ)}{}$



$= \tan(\overset{\alpha}{150^\circ} + \overset{\beta}{45^\circ}) = \dfrac{\tan 150^\circ + \tan 45^\circ}{1 - \tan 150^\circ \tan 45^\circ}$

expand w/ rule

QII

$= \dfrac{\left(-\frac{\sqrt{3}}{3}\right) + (1)}{1 - \left(-\frac{\sqrt{3}}{3}\right)(1)}$

LCD

$= \dfrac{3 \cdot 1 - \frac{\sqrt{3}}{\cancel{3}} \cdot \cancel{3}}{3 \cdot 1 + \frac{\sqrt{3}}{\cancel{3}} \cdot \cancel{3}}$

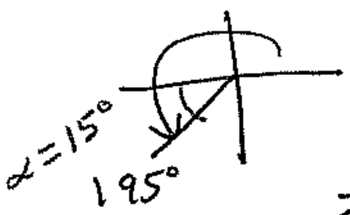
Conjugate

$= \dfrac{(3-\sqrt{3})}{(3+\sqrt{3})} \cdot \dfrac{(3-\sqrt{3})}{(3-\sqrt{3})} = \dfrac{9 - 3\sqrt{3} - 3\sqrt{3} + 3}{9 - \cancel{3\sqrt{3}} + \cancel{3\sqrt{3}} - 3} = \dfrac{12 - 6\sqrt{3}}{6}$

$= \boxed{2 - \sqrt{3}}$

Done another way

$$\tan 195^\circ = \overset{\text{Quadrant}}{+}\ \tan \overset{\text{reference angle}}{15^\circ}$$



S	A
T	C

$$= \tan(45^\circ - 30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{(1) - \left(\frac{\sqrt{3}}{3}\right)}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{3 \cdot 1 - \frac{\sqrt{3}}{\cancel{3}} \cdot \cancel{3}}{3 \cdot 1 + \frac{\sqrt{3}}{\cancel{3}} \cdot \cancel{3}}$$

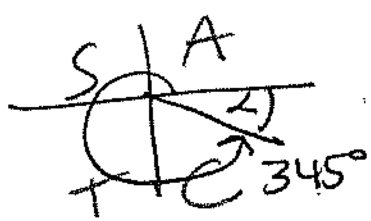
$$= \frac{(3 - \sqrt{3})}{(3 + \sqrt{3})} \cdot \frac{(3 - \sqrt{3})}{\underbrace{(3 - \sqrt{3})}_{\text{conjugate}}} = \frac{9 - 3\sqrt{3} - 3\sqrt{3} + 3}{9 - \cancel{3\sqrt{3}} + \cancel{3\sqrt{3}} - 3}$$

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= \frac{\cancel{6}(2 - \sqrt{3})}{\cancel{6}} = \boxed{2 - \sqrt{3}}$$

Quadrant $\quad$ reference angle

$$\tan 345^\circ = -\tan 15^\circ$$



$\measuredangle = 15^\circ$

S	A
T	C

$$= -\tan(45^\circ - 30^\circ)$$

$$= -\left[\frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\right]$$

$$= -\left[\frac{(1) - \left(\frac{\sqrt{3}}{3}\right)}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)}\right]$$

$$= -\left[\frac{3 \cdot 1 - \frac{\sqrt{3}}{3} \cdot 3}{3 \cdot 1 + \frac{\sqrt{3}}{3} \cdot 3}\right]$$

$$= -\left[\frac{(3 - \sqrt{3})}{(3 + \sqrt{3})} \frac{(3 - \sqrt{3})}{(3 - \sqrt{3})}\right]$$

$$= -\left[\frac{9 - 3\sqrt{3} - 3\sqrt{3} + 3}{9 - \cancel{3\sqrt{3}} + \cancel{3\sqrt{3}} - 3}\right]$$

$$= -\left[\frac{12 - 6\sqrt{3}}{6}\right] = -\left[\frac{\cancel{6}(2 - \sqrt{3})}{\cancel{6}}\right] = -(2 - \sqrt{3})$$

$$= \boxed{-2 + \sqrt{3}}$$

b) cot (−15°)

$30° - 45°$

$45° - 60°$

$= \cot(\overset{\alpha}{45°} - \overset{\beta}{60°})$

$= \dfrac{1 + \tan\alpha \tan\beta}{\tan\alpha - \tan\beta}$

$= \dfrac{1 + \tan 45° \tan 60°}{\tan 45° - \tan 60°}$

$= \dfrac{1 + (1)(\sqrt{3})}{(1) - (\sqrt{3})} = \dfrac{(1+\sqrt{3})}{(1-\sqrt{3})} \cdot \dfrac{(1+\sqrt{3})}{(1+\sqrt{3})}$ conjugate

$= \dfrac{1 + \sqrt{3} + \sqrt{3} + 3}{1 + \cancel{\sqrt{3}} - \cancel{\sqrt{3}} - 3} = \dfrac{4 + 2\sqrt{3}}{-2}$

$= \dfrac{(2 + \sqrt{3})}{-1} = \boxed{-2 - \sqrt{3}}$

For $\cot(\alpha - \beta)$

use $\tan(\alpha - \beta)$

$= \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

flipped to the reciprocal

Finding the Exact Value

Ex. 4: If $\sin\alpha = \dfrac{12}{13}$, $0 < \alpha < \dfrac{\pi}{2}$, and $\sin\beta = \dfrac{3}{5}$, $\dfrac{\pi}{2} < \beta < \pi$

a) Find cos(α + β)

★ you don't know the angles, so use Δ's and values.

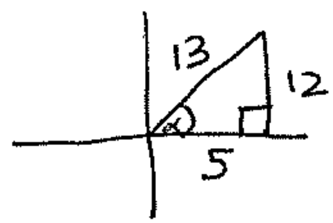
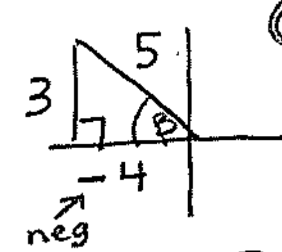
expand

$= \cos\alpha\cos\beta \overset{\text{opp}}{-} \sin\alpha\sin\beta$

$= \underbrace{\left(\frac{5}{13}\right)\left(-\frac{4}{5}\right)} - \underbrace{\left(\frac{12}{13}\right)\left(\frac{3}{5}\right)}$

$= \dfrac{-20}{65} - \dfrac{36}{65}$

$= \boxed{-\dfrac{56}{65}}$

α	β
QI	QII
	
$\sin\alpha = \frac{12}{13}$ SOH	$\sin\beta = \frac{3}{5}$
TRIPLET 5 − 12 − 13	TRIPLET 3 − 4 − 5
$\cos\alpha = \frac{5}{13}$ CAH	$\cos\beta = -\frac{4}{5}$ QII
$\tan\alpha = \frac{12}{5}$ TOA	$\tan\beta = -\frac{3}{4}$

S A
T C

b) Find $\tan(\alpha + \beta)$ for example 4.

expand

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\left(\frac{12}{5}\right) + \left(-\frac{3}{4}\right)}{1 - \underbrace{\left(\frac{12}{5}\right)\left(-\frac{3}{4}\right)}}$$

$$= \frac{\overset{4}{\cancel{20}} \cdot \frac{12}{\cancel{5}} - \frac{3}{\cancel{4}} \cdot \overset{5}{\cancel{20}}}{20 \cdot 1 + \frac{36}{\cancel{20}} \cdot \cancel{20}}$$

$$= \frac{48 - 15}{20 + 36} = \boxed{\frac{33}{56}}$$

Ex. 5: Find the value of the expression without a calculator.

$a)\ \cos\left[\underbrace{\cos^{-1}\left(-\frac{1}{2}\right)}_{\alpha}+\underbrace{\sin^{-1}1}_{\beta}\right]$

$= \cos(\alpha+\beta)$

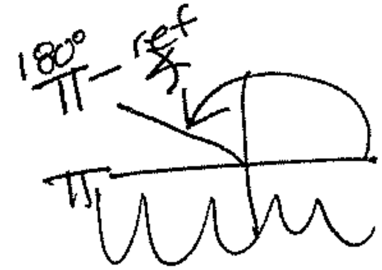
$= \cos(120^\circ+90^\circ)$

$= \cos(210^\circ)$

$= \pm\cos 30^\circ$

$= \boxed{-\frac{\sqrt{3}}{2}}$

$\alpha = \cos^{-1}\left(-\frac{1}{2}\right)$



ref ∡ $= \frac{\pi}{3}$ or 60°

$\alpha = \frac{2\pi}{3}$ or 120°

$\beta = \sin^{-1} 1$

QA value

$\beta = \frac{\pi}{2}$ or 90°

$$b)\ \tan\left(\overset{\alpha}{\sin^{-1}\frac{4}{5}} - \overset{\beta}{\cos^{-1}\frac{5}{13}}\right)$$

$$= \tan(\alpha - \beta)$$

Expand b/c we will not find these angles

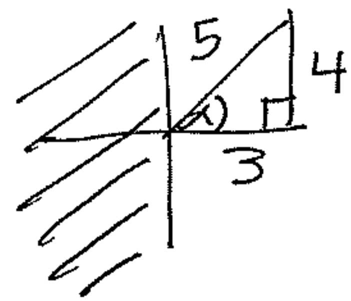
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\left(\frac{4}{3}\right) - \left(\frac{12}{5}\right)}{1 + \left(\frac{4}{3}\right)\left(\frac{12}{5}\right)}$$

$$= \frac{\overset{5}{\cancel{15}} \cdot \frac{4}{\cancel{3}} - \frac{12}{\cancel{5}} \cdot \overset{3}{\cancel{15}}}{15 \cdot 1 + \frac{48}{\cancel{15}} \cdot \cancel{15}}$$

$$= \frac{20 - 36}{15 + 48} = \boxed{\frac{-16}{63}}$$

$$\alpha = \sin^{-1}\frac{4}{5}$$



so ...

$$\sin\alpha = \frac{4}{5}$$

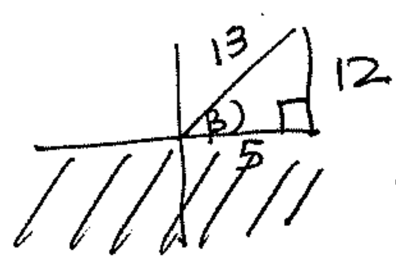
SOH

TRIPLET 3-4-5

$$\tan\alpha = \frac{4}{3}$$

TOA

$$\beta = \cos^{-1}\frac{5}{13}$$



so ...

$$\cos\beta = \frac{5}{13}$$

CAH

TRIPLET 5-12-13

$$\tan\beta = \frac{12}{5}$$

The sum/diff. formulas can be used to verify many identities that we have seen, such as the cofunction rule sin (90° – θ) = cos θ, and to derive new identities.

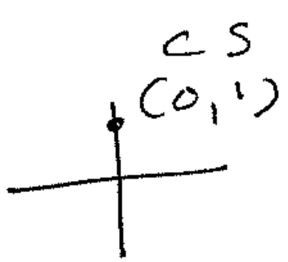
Ex. 6: Verify the identity: $\sec\left(\overset{\alpha}{\frac{\pi}{2}} - \overset{\beta}{x}\right) = \boxed{\csc x}$ ✓

$$\frac{\cos(\alpha - \beta)}{1}$$

Take reciprocal

$$\frac{1}{\cos(\overset{\alpha}{\frac{\pi}{2}} - \overset{\beta}{x})} =$$

now expand



$$\frac{1}{\cos\frac{\pi}{2}\cos x \underset{opp}{+} \sin\frac{\pi}{2}\sin x} =$$

$$\frac{1}{\cancel{(0)\cos x} + (1)\sin x} =$$

0

$$\frac{1}{\sin x} =$$

✓ $\boxed{\csc x} =$

Ex. 7: Write the trigonometric expression as an algebraic expression.

$$\cos(\underbrace{\arccos x}_{\alpha} - \underbrace{\arcsin x}_{\beta})$$

inverse

$= \cos(\alpha - \beta)$

expand (don't know angles)

$= \cos\alpha \cos\beta + \sin\alpha \sin\beta$

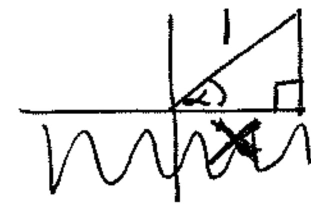
opp

$= \underbrace{(x)(\sqrt{1-x^2})} + \underbrace{(\sqrt{1-x^2})(x)}$

$= 1x\sqrt{1-x^2} + 1x\sqrt{1-x^2}$

$= \boxed{2x\sqrt{1-x^2}}$

$\alpha = \arccos x$



so... $\cos\alpha = \dfrac{x}{1}$

CAH

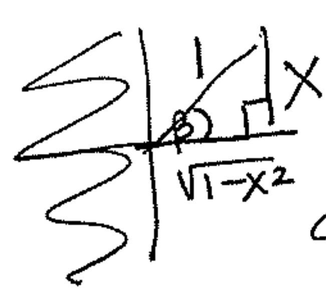
$a^2 + x^2 = (1)^2$

$a^2 = 1 - x^2$

$a = \sqrt{1-x^2}$

$\sin\alpha = \dfrac{\sqrt{1-x^2}}{1}$

$\beta = \arcsin x$



so... $\sin\beta = \dfrac{x}{1}$

SOH

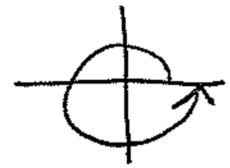
$\cos\beta = \dfrac{\sqrt{1-x^2}}{1}$

CAH

Find "X"

Ex. 8: Find the solutions of the equation in the interval $[0, 2\pi)$

Radians



$$\cos\left(\overset{\alpha}{x} + \overset{\beta}{\frac{\pi}{6}}\right) - \cos\left(\overset{\alpha}{x} - \overset{\beta}{\frac{\pi}{6}}\right) = 1$$

expand expand

$$\cos x \cos\frac{\pi}{6} \underset{\text{opp}}{-} \sin x \sin\frac{\pi}{6} - \left(\cos x \cos\frac{\pi}{6} \underset{\text{opp}}{+} \sin x \sin\frac{\pi}{6}\right) = 1$$

$$\cancel{\cos x \cos\frac{\pi}{6}} - \sin x \sin\frac{\pi}{6} - \cancel{\cos x \cos\frac{\pi}{6}} - \sin x \sin\frac{\pi}{6} = 1$$

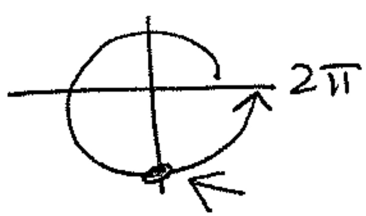
$$-2 \sin x \sin\frac{\pi}{6} = 1$$

$$\sin x \sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\sin x \left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$\cancel{2} \cdot \frac{1}{\cancel{2}} \sin x = -\frac{1}{\cancel{2}} \cdot \cancel{2}$$

$$\sin x = -1$$



$$\boxed{x = \frac{3\pi}{2}}$$