Notes

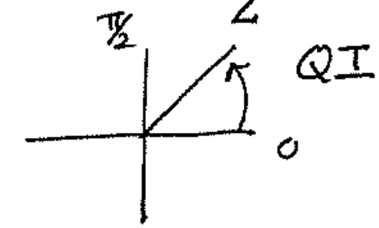
Pre-Calculus: Sec. 5.1
Trigonometric Identities (Day 2)

Ex.1)Use a trigonometric substitution to write the algebraic expression as a trig. function of θ.

1a) $\sqrt{64-16x^2}$, $x = 2\cos\theta$ where $0 < \theta < \frac{\pi}{2}$.

$(x)^2 = (2\cos\theta)^2$

$x^2 = 4\cos^2\theta$



$\sqrt{64 - 16(4\cos^2\theta)}$

$\sqrt{64 - 64\cos^2\theta}$

factor $\sqrt{64(1-\cos^2\theta)}$

$8\sqrt{1-\cos^2\theta}$ ← pyth

$8\sqrt{\sin^2\theta}$

$8|\sin\theta|$

EVEN-EVEN-ODD

$\sqrt[2]{x^2} = |x^1|$

Quad I

S	A
T	C

All positive in Quad I

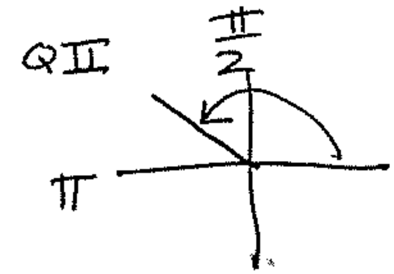
$= \boxed{8\sin\theta}$

Decide whether to keep the absolute value based on the quadrant.

1b) $\sqrt{\underline{\underline{9x^2}}+4}$, $3x = 2\tan\theta$ where $\frac{\pi}{2} < \theta < \pi$.

$(3x)^2 = (2\tan\theta)^2$

$9x^2 = 4\tan^2\theta$



$\sqrt{4\tan^2\theta + 4}$

factor $\sqrt{4(\tan^2\theta + 1)}$

$2\sqrt{\tan^2\theta + 1}$ ← pyth

$2\sqrt{\sec^2\theta}$

$\boxed{2|\sec\theta|}$ Keep absolute value?

QII → S | A
T | C

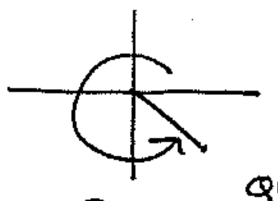
$\underline{\underline{\text{yes}}}$ for Quad II.

1c)

Use $a = 7\cos\theta$ to rewrite $\dfrac{1}{\sqrt{49-a^2}}$ as a trigonometric function involving θ,

where $\dfrac{3\pi}{2} < \theta < 2\pi$.

$$(a)^2 = (7\cos\theta)^2$$
$$a^2 = 49\cos^2\theta$$



$$\frac{1}{\sqrt{49-49\cos^2\theta}}$$

factor

$$\frac{1}{\sqrt{49(1-\cos^2\theta)}}$$

$$\frac{1}{7\sqrt{1-\cos^2\theta}} = \frac{1}{7\sqrt{\sin^2\theta}} =$$

Pyth

S | A
T | C

Quad IV

$$\boxed{\frac{1}{7|\sin\theta|}} = \boxed{\frac{|\csc\theta|}{7}} = \boxed{\frac{1}{7}|\csc\theta|}$$

need abs. value ?

yes

Pythagorean Identities:

$$cos^2\theta + sin^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$cot^2\theta + 1 = csc^2\theta$$

angles add up to 90° or $\frac{\pi}{2}$

Ex.2) Use *cofunction* identities to evaluate without a calculator.

$a)\ \cos^2 14° + \cos^2 76°$

$\sin^2(90° - 14°)$

$\sin^2 76° + \cos^2 76° = \boxed{1}$

pyth. identity

★ Cofunction rules apply to any power.

$b)\ \sin^2 18° + \sin^2 40° + \sin^2 50° + \sin^2 72°$

$\cos^2(90° - 18°) + \cos^2(90° - 40°)$

$\cos^2 72° + \sin^2 72° + \cos^2 50° + \sin^2 50°$

$1 + 1 = \boxed{2}$

Ex. 3) Simplify: $\left(\cos^2 x - 1\right)\left(1 + \cot^2 x\right)$

w/ pyth

$$-\sin^2 x \cdot \csc^2 x$$

$$-\cancel{\sin^2 x} \cdot \frac{1}{\cancel{\sin^2 x}}$$

$$\boxed{-1}$$

Ex.4) Verify the following identities:

a) $$\frac{1+\csc\theta}{\cot\theta+\cos\theta} = \sec\theta$$

mult. by the LCD

$$\frac{\sin\theta\cdot\frac{1}{1}+\frac{1}{\cancel{\sin\theta}}\cdot\cancel{\sin\theta}}{\cancel{\sin\theta}\cdot\frac{\cos\theta}{\cancel{\sin\theta}}+\frac{\cos\theta}{1}\cdot\sin\theta} = \boxed{\frac{1}{\cos\theta}}$$ ✓

$$\frac{\sin\theta+1}{\cos\theta+\cos\theta\sin\theta} =$$

factor →

$$\frac{\cancel{\sin\theta+1}^{\,1}}{\cos\theta(\cancel{1+\sin\theta})} =$$

✓ $$\boxed{\frac{1}{\cos\theta}} =$$

b) $$\frac{\cot^2\theta}{1+\csc\theta} = \frac{1-\sin\theta}{\sin\theta}$$

pyth

★ one term (split up)

factor DOS

$$\frac{\csc^2\theta - 1}{1+\csc\theta} =$$

$$\frac{\cancel{(\csc\theta+1)}(\csc\theta-1)}{\cancel{1+\csc\theta}} = \frac{1}{\sin\theta} - \frac{\cancel{\sin\theta}}{\sin\theta}$$

$$\boxed{\csc\theta - 1} \checkmark = \boxed{\csc\theta - 1} \checkmark$$

or

$$\frac{1}{\sin\theta} - 1 =$$

need common denom

$$\frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta} =$$

matches top right

$$\boxed{\frac{1-\sin\theta}{\sin\theta}} =$$

c) $\cos^2\theta - \underline{\sin^2\theta} = \boxed{2\cos^2\theta - 1}$

$\cos^2\theta - (1 - \cos^2\theta) =$

$\underline{\cos^2\theta} - 1 + \underline{\cos^2\theta} =$

$\boxed{2\cos^2\theta - 1} =$

d) $1 + \underline{\sec^2\theta} = \boxed{2 + \tan^2\theta}$

$1 + (1 + \tan^2\theta) =$

$\boxed{2 + \tan^2\theta} =$

e) $\sec^4 x \tan^2 x = \boxed{(\tan^2 x + \tan^4 x)\sec^2 x}$

$\sec^2 x \cdot \underline{\underline{\sec^2 x}} \cdot \tan^2 x =$

$\sec^2 x\,(1 + \tan^2 x) \cdot \tan^2 x =$

$\sec^2 x\,(\tan^2 x + \tan^4 x) =$

reorder

$\boxed{(\tan^2 x + \tan^4 x)\sec^2 x} =$

Done another way

e) $\boxed{\sec^4 x \tan^2 x} = (\tan^2 x + \tan^4 x)\sec^2 x$

factor gcf

$= \tan^2 x\,(\underbrace{1+\tan^2 x}_{\text{pyth}})\,\sec^2 x$

$= \tan^2 x \cdot \sec^2 x \cdot \sec^2 x$

$= \tan^2 x \cdot \sec^4 x$

reorder

$= \boxed{\sec^4 x \cdot \tan^2 x}$ ✓

Done another way

e) $\sec^4 x \underbrace{\tan^2 x}_{\text{pyth}} = (\tan^2 x + \tan^4 x)\sec^2 x$

$\sec^4 x(\sec^2 x - 1) = (\tan^2 x + \tan^2 x \cdot \tan^2 x)\sec^2 x$

$\boxed{\sec^6 x - \sec^4 x}$ ✓ $= [\sec^2 x - 1 + \underbrace{(\sec^2 x - 1)(\sec^2 x - 1)}_{\text{Foil}}]\sec^2 x$

$= (\sec^2 x - \cancel{1} + \sec^4 x - 2\sec^2 x + \cancel{1})\sec^2 x$

combine like terms

$= (\sec^4 x - \sec^2 x)\sec^2 x$

$= \boxed{\sec^6 x - \sec^4 x}$ ✓

f) $\sin^4 x + \cos^4 x = \boxed{1 - 2\cos^2 x + 2\cos^4 x}$

$\sin^2 x \cdot \sin^2 x + \cos^4 x =$

$(1-\cos^2 x)(1-\cos^2 x) + \cos^4 x =$

FOIL

$1 - 2\cos^2 x + \underline{\cos^4 x} + \underline{\cos^4 x} =$

$\boxed{1 - 2\cos^2 x + 2\cos^4 x} =$

Pre-Calculus
Lesson 3
Sec. 5.2
Sum and Difference Formulas

The Sum and Difference Formula for Sine

• Same sign in a sine sandwich

Sum: $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Difference: $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

Condense $\longleftrightarrow$ Expand

The Sum and Difference Formula for Cosine

Sum: $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Difference: $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

The Two Main Purposes for the Sum/Diff. Formulas:

1)Finding the **exact** values of trigonometric expressions with non common angles.

2)Simplifying expressions to obtain other identities.

non calc

Ex. 1: Find the exact value of each of the following:

$a)\ \cos 15^\circ$

$60^\circ - 45^\circ$

$45^\circ - 30^\circ$

$= \cos(\overset{\alpha}{45^\circ} - \overset{\beta}{30^\circ})$

Expand w/ rule

$= \cos\alpha \cos\beta \overset{opp}{+} \sin\alpha \sin\beta$

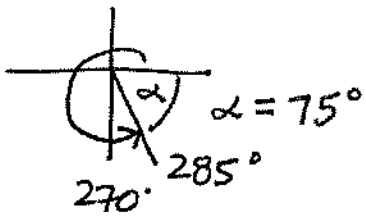
$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}_{\frac{\sqrt{6}}{4}} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)}_{\frac{\sqrt{2}}{4}}$

$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$$b)\ \sin 285^\circ = \overset{\text{Quadrant}}{-}\ \sin \overset{\text{reference angle}}{75^\circ}$$



S	A
T	C

$$= -\left[\sin(30^\circ + 45^\circ)\right]$$

expand w/ rule

$$= -\left[\sin 30^\circ \cos 45^\circ \overset{\text{same}}{+} \cos 30^\circ \sin 45^\circ\right]$$

$$= -\left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\left[\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right]$$

$$= \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$$

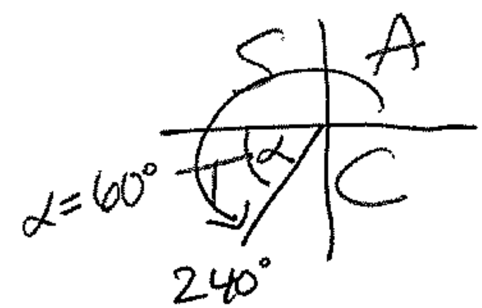
Done another way

b) $\sin 285°$

$45° + \underset{(60°)}{240°}$

$\underset{(45°)}{315°} - 30°$

$\underset{(45°)}{225°} + 60°$



$= \sin(\overset{\alpha}{45°} + \overset{\beta}{240°})$

Expand w/rule

$= \sin\alpha\cos\beta \overset{\text{same}}{+} \cos\alpha\sin\beta$

$= \sin 45° \cos 240° + \cos 45° \sin 240°$

$= \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)}_{-\frac{\sqrt{2}}{4}} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)}_{\left(-\frac{\sqrt{6}}{4}\right)}$

$-\frac{\sqrt{2}}{4} \underbrace{+ \left(-\right.}_{\text{combine signs}}\left.\frac{\sqrt{6}}{4}\right) = \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$

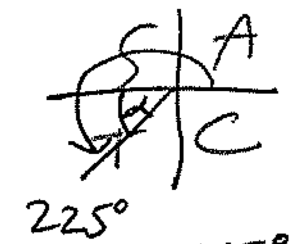
Done another way

b) $\sin 285°$

$$45° + \underset{(60°)}{240°}$$

$$\underset{(45°)}{315°} - 30°$$

$$\underset{(45°)}{225°} + 60°$$



$\alpha = 45°$

QIII

$$= \sin(\overset{\alpha}{225°} + \overset{\beta}{60°})$$

Expand w/ rule

$$= \sin\alpha\cos\beta \overset{\text{same}}{+} \cos\alpha\sin\beta$$

$$= \sin 225°\cos 60° + \cos 225°\sin 60°$$

$$= \underbrace{\left(\overset{\text{Quad III}}{-}\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)}_{-\frac{\sqrt{2}}{4}} + \underbrace{\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}_{\left(-\frac{\sqrt{6}}{4}\right)}$$

$$-\frac{\sqrt{2}}{4} \underbrace{+ \ (}_{\text{Combine signs}} -\frac{\sqrt{6}}{4}) = \boxed{\frac{-\sqrt{2}-\sqrt{6}}{4}}$$

Ex.2: Simplify each of the following expressions.

Can you condense using a sum/difference formula?

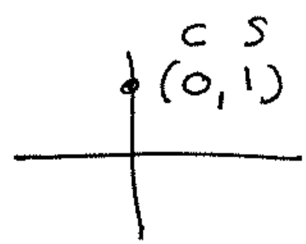
a) $\underbrace{\cos 50^\circ}_{\alpha} \underbrace{\cos 40^\circ}_{\beta} \underbrace{-}_{\downarrow} \underbrace{\sin 50^\circ}_{\alpha} \underbrace{\sin 40^\circ}_{\beta}$ Pattern?

$= \cos(\alpha \underset{\text{opp}}{+} \beta)$

$= \cos(50^\circ + 40^\circ)$

$= \cos 90^\circ$

$= \boxed{0}$



b) $\sin\overset{\alpha}{\frac{11\pi}{30}}\cos\overset{\beta}{\frac{\pi}{5}} - \underbrace{\sin\overset{\beta}{\frac{\pi}{5}}\cos\overset{\alpha}{\frac{11\pi}{30}}}_{\text{reorder}}$

Pattern?

Sine Sandwich!

$$\sin\overset{\alpha}{\frac{11\pi}{30}}\cos\overset{\beta}{\frac{\pi}{5}} - \cos\overset{\alpha}{\frac{11\pi}{30}}\sin\overset{\beta}{\frac{\pi}{5}}$$

same

$$= \sin(\alpha - \beta)$$

$$= \sin\left(\frac{11\pi}{30} - \frac{\pi}{5}\right)$$ get common denom.

$$= \sin\left(\frac{11\pi}{30} - \frac{6\pi}{30}\right)$$

$$= \sin\left(\frac{5\pi}{30}\right)$$

$$= \sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$$