Notes

# Pre-Calculus: Sec. 5.1 Trigonometric Identities

Reciprocal Identities:

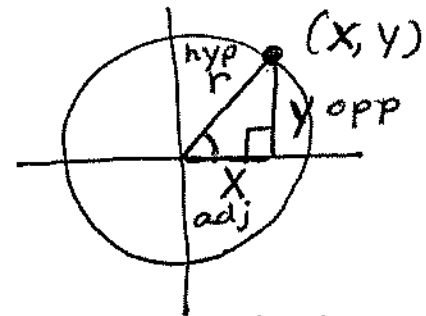
$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities (Dividing two trig. functions.):

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

# From the Pythagorean Theorem:




$$a^2 + b^2 = c^2$$

$$x^2 + y^2 = r^2 \text{ (circle)}$$

$$x^2 + y^2 = 1 \text{ (unit circle)}$$

CAH SOH $r=1$ TOA

since $\boxed{\cos\theta = \frac{x}{r}}$ and $\boxed{\sin\theta = \frac{y}{r}}$ and in a unit circle r = 1 $\boxed{\tan\theta = \frac{y}{x}}$

Then.....$x^2 + y^2 = 1$ can be written as:

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\cos^2\theta + \sin^2\theta = 1$

$\cos\theta^2$ squares theta here

on the unit ⊙

$\cos\theta = \frac{x}{1}$ $\sin\theta = \frac{y}{1}$

$\cos\theta = x$ $\sin\theta = y$

Pythagorean Identity #1:

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta - 1 = -\sin^2\theta \quad \text{etc...}$$

# More from the Pythagorean Theorem

From: $\cos^2\theta + \sin^2\theta = 1$, we can derive two others…

## Pythagorean Identity #2:

divide by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$$1 = \sec^2\theta - \tan^2\theta$$

$$\tan^2\theta - \sec^2\theta = -1$$

etc…

## Pythagorean Identity #3:

divide by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{\cot^2\theta + 1 = \csc^2\theta}$$

Pythagorean Identities:

$$cos^2\theta + sin^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$cot^2\theta + 1 = csc^2\theta$$

Ex.1) Simplify the expression:

a) $\sin x(\csc x - \sin x)$

$\sin x \cdot \csc x - \sin^2 x$

$\frac{\cancel{\sin x}}{1} \cdot \frac{1}{\cancel{\sin x}} - \sin^2 x$

pyth. $1 - \sin^2 x$

$= \boxed{\cos^2 x}$

b) $\underbrace{\cot(\pi/2 - x)}_{\text{cofunction}}\cos x$

$\tan x \cdot \cos x$

$\frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1}$

$= \boxed{\sin x}$

Ex.2) Factor and simplify the expression.

gcf $\rightarrow$ $\underline{\underline{\sec^2 x}}\tan^2 x + \underline{\underline{\sec^2 x}}$

$$\sec^2 x\,(\underbrace{\tan^2 x + 1}_{\text{pyth}})$$

$$\sec^2 x \cdot \sec^2 x$$

$$= \boxed{\sec^4 x}$$

Ex.3) Simplify the expression.

$$\sec^2 x\,\underbrace{(1-\sin^2 x)}_{\text{pyth}}$$

$$\sec^2 x \cdot \cos^2 x$$

$$\frac{1}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{1}$$

$$= \boxed{1}$$

# Guidelines for Proving Trigonometric Identities

- **Work** with each side of the equation independently.
- **Start** with the more complicated side and **transform** it step–by–step until both sides look the same.
- **Look** for opportunities to **apply** the **identities**.
- **Rewriting** the more complicated side of the equation in terms of **sine** and **cosine** is often helpful. But NOT always!
- **Simplify** algebraically: factor, combine (common denominator) 2 fractions, distribute (multiply together), ….ect.
- You may **not move across** the = or × and ÷ on both sides. Do not try to "balance."
- Always keep this in mind "Your answer is already there!!!"

Ex. 4) Prove (Transform one side of the equation into the other.)

need common denom.

$$\frac{1}{(1-\sec\theta)}+\frac{1}{(1+\sec\theta)} = \boxed{-2\cot^2\theta}$$

$$\frac{(1+\sec\theta)}{(1+\sec\theta)}\cdot\frac{1}{(1-\sec\theta)} + \frac{1}{(1+\sec\theta)}\cdot\frac{(1-\sec\theta)}{(1-\sec\theta)} =$$

$$\frac{1+\cancel{\sec\theta}+1-\cancel{\sec\theta}}{(1+\sec\theta)(1-\sec\theta)} =$$

FOIL

$$\frac{2}{1-\sec^2\theta} =$$

Pyth

$$\frac{2}{-\tan^2\theta} =$$

$$-2\cdot\frac{1}{\tan^2\theta} =$$

$$\boxed{-2\cot^2\theta} =$$

Ex. 5) Prove (Verify the identity algebraically.)

Cofunction →

$$\frac{\cos x}{\cos\left(\frac{\pi}{2} - x\right)} = \boxed{\cot x} ✓$$

$$\frac{\cos x}{\sin x} =$$

$$\boxed{\cot x} = ✓$$

## Ex.6) Prove:

Foil

$$(\sec\theta - \tan\theta)(\csc\theta + 1) = \boxed{\cot\theta}$$

$$\sec\theta\csc\theta + \sec\theta - \tan\theta\csc\theta - \tan\theta =$$

$$\underbrace{\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}} + \frac{1}{\cos\theta} - \underbrace{\frac{\cancel{\sin\theta}}{\cos\theta}\cdot\frac{1}{\cancel{\sin\theta}}} - \frac{\sin\theta}{\cos\theta} =$$

$$\frac{1}{\cos\theta\sin\theta} + \cancel{\frac{1}{\cos\theta}} - \cancel{\frac{1}{\cos\theta}} - \frac{\sin\theta}{\cos\theta} =$$

need common denom. $\quad \frac{1}{\cos\theta\sin\theta} - \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta} =$

pyth $\rightarrow \quad \frac{1 - \sin^2\theta}{\cos\theta\sin\theta} =$

$$\frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta}\sin\theta} =$$

$$\frac{\cos\theta}{\sin\theta} =$$

✓ $\boxed{\cot\theta} =$

## Ex. 7) Prove:

$$\frac{\tan\theta}{(1+\sec\theta)}+\frac{(1+\sec\theta)}{\tan\theta} = \boxed{2\csc\theta}$$

✓

need common denom.

$$\frac{\tan\theta}{\tan\theta}\cdot\frac{\tan\theta}{(1+\sec\theta)} + \frac{(1+\sec\theta)}{\tan\theta}\cdot\frac{(1+\sec\theta)}{(1+\sec\theta)} =$$

FOIL

$$\frac{\tan^2\theta + (1+\sec\theta)(1+\sec\theta)}{\tan\theta\,(1+\sec\theta)} =$$

pyth $\rightarrow$ $$\frac{\tan^2\theta + 1 + 2\sec\theta + \sec^2\theta}{\tan\theta\,(1+\sec\theta)} =$$

combine like terms $\rightarrow$ $$\frac{\sec^2\theta + 2\sec\theta + \sec^2\theta}{\tan\theta\,(1+\sec\theta)} =$$

gcf $\rightarrow$ $$\frac{2\sec^2\theta + 2\sec\theta}{\tan\theta\,(1+\sec\theta)} =$$

$$\frac{2\sec\theta\,\cancel{(\sec\theta+1)}}{\tan\theta\,\cancel{(1+\sec\theta)}} =$$

Left side continued

$$\frac{2\sec\theta}{\tan\theta} =$$

$$\frac{2\cdot\frac{1}{\cancel{\cos\theta}}}{\frac{\sin\theta}{\cancel{\cos\theta}}} =$$

$$\frac{2}{\sin\theta} =$$

✓ $\boxed{2\csc\theta}$ =

Ex. 8) Prove:

cofunction

odd function

$$\frac{\csc(\frac{\pi}{2}-\theta)}{\tan(-\theta)} = \boxed{-\csc\theta}$$

✓

$$\frac{\sec\theta}{-\tan\theta} =$$

$$\frac{\frac{1}{\cancel{\cos\theta}}}{-\frac{\sin\theta}{\cancel{\cos\theta}}}$$

$$-\frac{1}{\sin\theta} =$$

✓ $$\boxed{-\csc\theta} =$$

Ex. 9) Prove: Factor by grouping

$$\sec^3 x - \sec^2 x - \sec x + 1 = \boxed{(\sec x - 1)\tan^2 x}$$

$$\sec^2 x(\sec x - 1) - 1(\sec x - 1) =$$

$$(\sec x - 1)(\underbrace{\sec^2 x - 1}_{\text{Pyth}}) =$$

$$\boxed{(\sec x - 1)\cdot \tan^2 x}$$

or

$$\downarrow \text{pyth}$$

$$(\sec x - 1)(\sec^2 x - 1)$$

Foil

$$\sec^3 x - \sec x - \sec^2 x + 1$$

reorder

$$\boxed{\sec^3 x - \sec^2 x - \sec x + 1}$$

matches top left

Ex. 10) Prove:

$$\sec^4 x - \tan^4 x \quad = \quad \boxed{2\sec^2 x - 1}$$

DOS
(difference of squares)

$$(\sec^2 x + \tan^2 x)\underbrace{(\sec^2 x - \tan^2 x)}_{\text{pyth}} =$$

$$(\sec^2 x + \tan^2 x) \cdot 1 =$$

$$\sec^2 x + \tan^2 x =$$

pyth

combine like terms

$$\sec^2 x + (\sec^2 x - 1) =$$

$$\boxed{2\sec^2 x - 1} =$$

Ex.11) Prove. Hint: Use the conjugate.

$$\frac{5}{tanx+secx} = \boxed{5(secx - tanx)}$$

✓

$$\frac{5}{(tanx+secx)} \cdot \underbrace{\frac{(tanx-secx)}{(tanx-secx)}}_{\text{conjugate}} =$$

$$\frac{5(tanx-secx)}{tan^2x - \cancel{tanxsecx} + \cancel{tanxsecx} - sec^2x} =$$

pyth →

$$\frac{5(tanx-secx)}{tan^2x - sec^2x} =$$

$$\frac{5(tanx-secx)}{-1} =$$

$$-5(tanx-secx) =$$

$$5(-tanx+secx) =$$

✓ $\boxed{5(secx-tanx)} =$

Ex.12) Prove.

$\ln \to \log_e$

↗ Base e

$$\ln|\csc\theta| + \ln|\tan\theta| = \boxed{\ln|sec\theta|}$$

add

condenses

$$\ln|\csc\theta \cdot \tan\theta| =$$

multiply

$$\ln\left|\frac{1}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{\cos\theta}\right| =$$

$$\ln\left|\frac{1}{\cos\theta}\right| =$$

$$\boxed{\ln|\sec\theta|} =$$