Sec. 9.5 Parametric Equations

Parametric Equations: equations that express both x and y as functions of t (time). Parametric equations of a line are useful in finding the times and positions at which an object moving with constant velocity crosses a curve (line, etc...) whose equation is known. They are useful for modeling the path of an object.

The parameter, t, will allow us to compute points.
(or θ, or any variable)

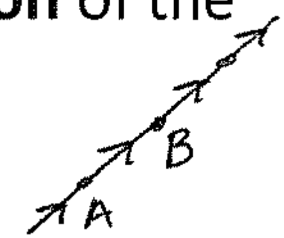
The set of ordered pairs (x, y) that are generated from the parametric equations is called a **Plane Curve**.

Graphing a plane curve represented by parametric equations involves plotting points in the rectangular coordinate system and connecting them with a smooth curve.

order matters

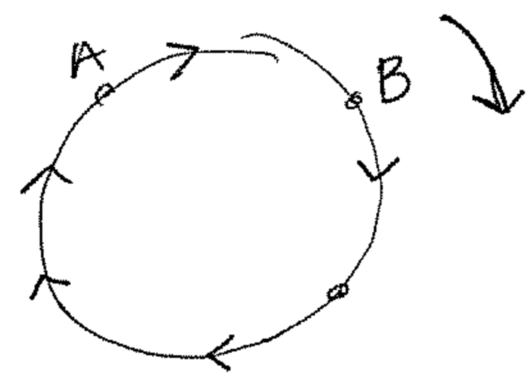
The points should be plotted in the order of increasing t values. This will tell us the direction, or ***orientation*** of the curves as t increases. We use **arrows** between the points to **show the orientation** of the curve corresponding to increasing values of t.

w/out the arrowheads the plane curve is incomplete.



t, θ, etc....

The process of **eliminating the parameter** finds a rectangular equation (an equation in terms of x and y only) that has the same curve as the parametric equations.



Sketch the curve represented by the parametric equations and write the corresponding rectangular equation by eliminating the parameter.

Eliminate the parameter (t):

Ex.1: $x = t - 1$

$y = t^2$

(t) domain all reals

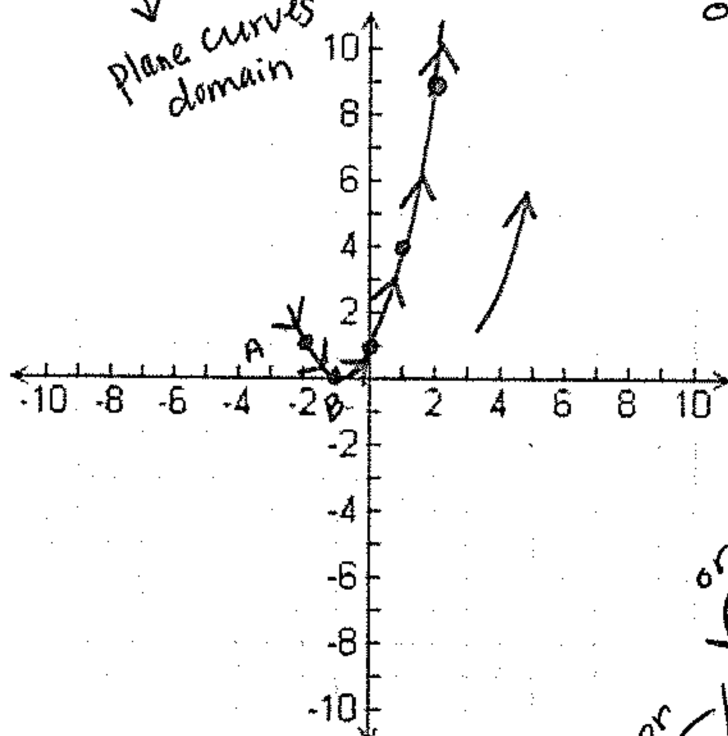
Range: $x \in (-\infty, \infty)$

decides if a restriction needs to be added to the rectangular equation

plane curve's domain

order t	x	y	
-1	(-2,	1)	A
0	(-1,	0)	B
1	(0,	1)	
2	(1,	4)	
3	(2,	9)	

increase



$x = t - 1$ $\quad x \in (-\infty, \infty)$

$y = t^2$

no domain restriction on rectangular equation

- Solve for t in one equation ...

$x = t - 1$

$x + 1 = t$

- Substitute into the other equation ...

$y = t^2$

$\boxed{y = (x+1)^2}$ $\boxed{\text{Parabola}}$

or

$\boxed{(x+1)^2 = y}$

or

$\boxed{y = x^2 + 2x + 1}$

logical for graph? yes #

Ex.2: $x = 3 - 2t$

$y = 2 + 3t$

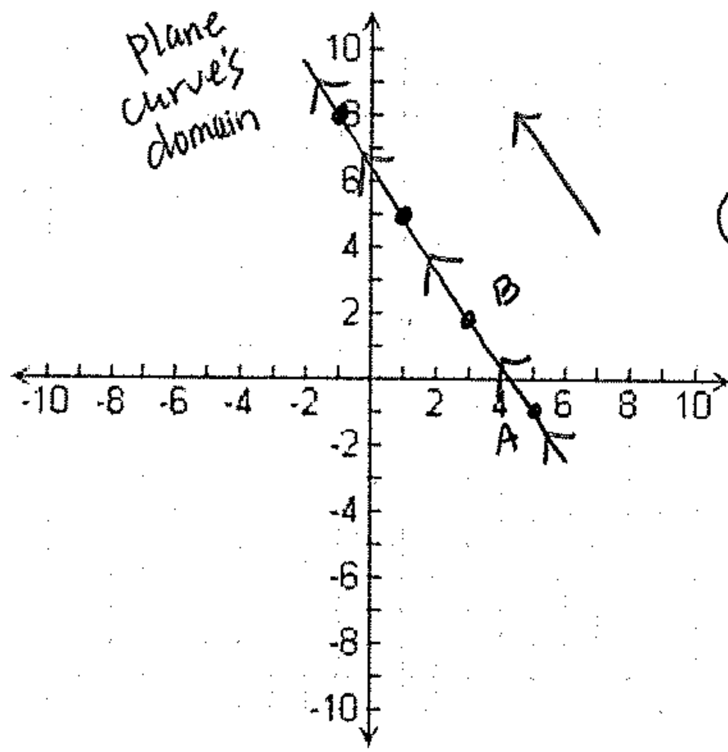
(t) domain all reals

range: $x \in (-\infty, \infty)$

t	x	y	
-1	$(5,$	$-1)$	A
0	$(3,$	$2)$	B
1	$(1,$	$5)$	
2	$(-1,$	$8)$	

increase

plane curve's domain



Eliminate Parameter (t)

$x = 3 - 2t \quad x \in (-\infty, \infty)$, no domain restriction on the rectangular equation

$y = 2 + 3t$

- Solve for t in one...

$x = 3 - 2t$

$\frac{x-3}{-2} = \frac{-2t}{-2}$

$-\frac{x}{2} + \frac{3}{2} = t$

- Substitute into the other equation...

$y = 2 + 3t$

$y = 2 + 3\left(-\frac{x}{2} + \frac{3}{2}\right)$

$y = \frac{2}{1} - \frac{3x}{2} + \frac{9}{2}$

$\boxed{y = \frac{-3x}{2} + \frac{13}{2}}$ $\boxed{\text{Linear}}$

$y = mx + b$

logical?

OR

$2\left(y = \frac{-3x}{2} + \frac{13}{2}\right)$

$2y = -3x + 13$

$\boxed{3x + 2y = 13}$

$\boxed{3x + 2y - 13 = 0}$

Caution: when converting from parametric to rectangular form, the range of x and y implied by the parametric equations may be altered by the change of the rectangular form.

add a domain restriction?

So…you may need to change the domain of the rectangular equation to be consistent with the domain for the parametric equation in x.

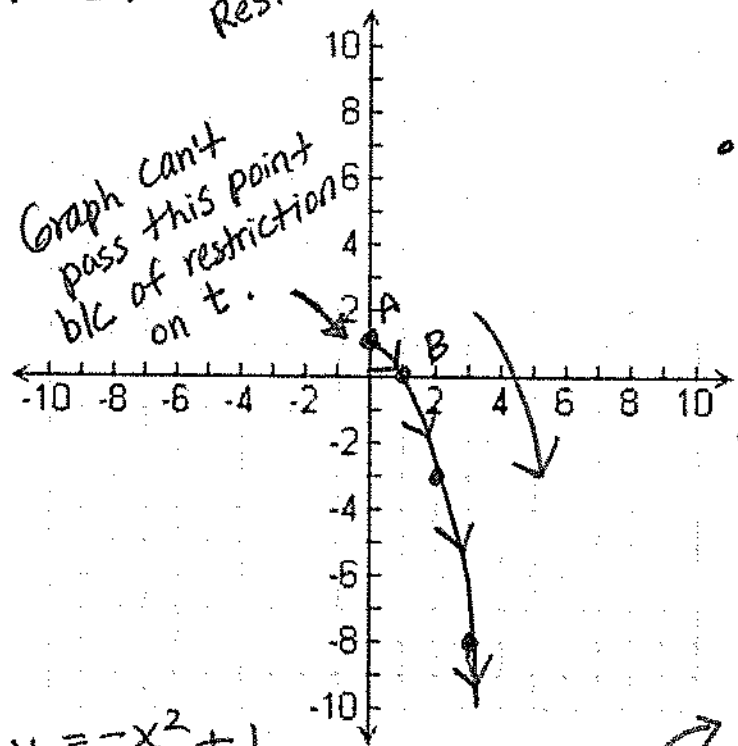
Ex.3: $x = \sqrt{t}$

$y = 1 - t$

(t) domain $t \geq 0$

t	x	y	
0	(0,	1)	A
1	(1,	0)	B
4	(2,	-3)	
9	(3,	-8)	

range $x \in [0, \infty)$ Restriction?



$y = -x^2 + 1$

$x^2 = -y + 1$

$\boxed{x^2 = -(y-1) : x \geq 0}$

or

Eliminate the parameter (t):

$x = \sqrt{t}$ $\quad$ $x \in [0, \infty)$ add this restriction to the rectangular equation

$y = 1 - t$

- Solve for t in one:

$(x)^2 = (\sqrt{t})^2$

$x^2 = t$

- Substitute into the other:

$y = 1 - t$

$y = 1 - x^2$

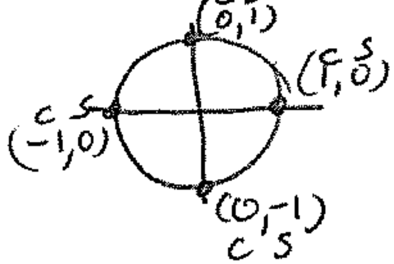
$\boxed{y = -x^2 + 1 ; \ x \geq 0}$

add on restriction (matches graph)

$\boxed{\text{Parabola}}$ w/restriction

When you have a **trigonometric equation**, you will need to use another process to eliminate the parameter…One method is to **square both sides** and **add** or **subtract** the sides of each equation together. (You are attempting to form any of the Pythagorean Identities).

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$



Eliminate the Parameter: (θ)

Ex.4: $x = 2\sin\theta$ range: $x \in [-2,2]$ restriction

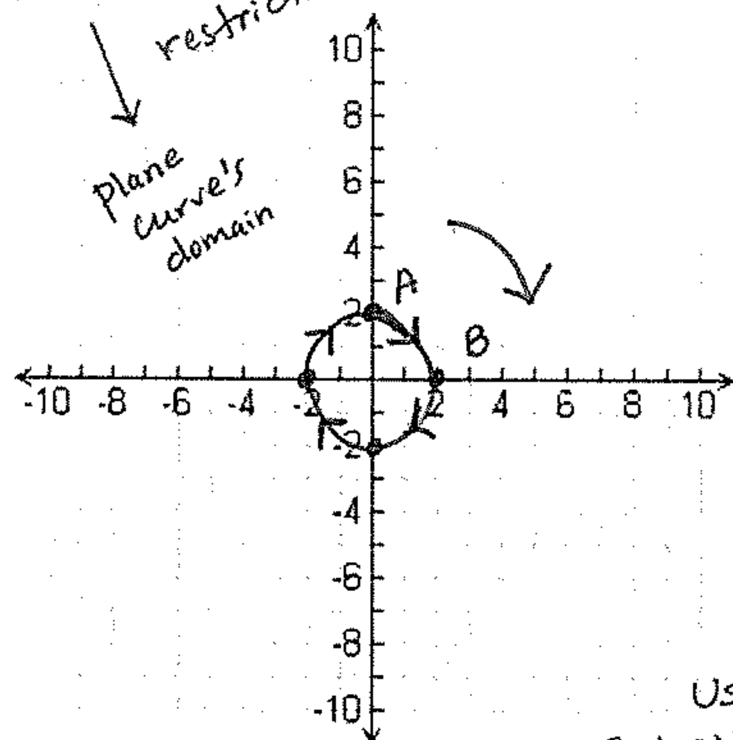
$y = 2\cos\theta$

θ domain all reals

increase

θ	x	y	
0	(0,	2)	A
$\frac{\pi}{2}$	(2,	0)	B
π	(0,	-2)	
$\frac{3\pi}{2}$	(-2,	0)	

plane curve's domain



$x = 2\sin\theta$ $\quad$ $y = 2\cos\theta$

$x \in [-2,2]$

I need to generate squared trig. functions to use the pythagorean identities…

x	y
$(x)^2 = (2\sin\theta)^2$	$(y)^2 = (2\cos\theta)^2$
$x^2 = 4\sin^2\theta$	$y^2 = 4\cos^2\theta$
$\frac{x^2}{4} = \sin^2\theta$	$\frac{y^2}{4} = \cos^2\theta$

Use: $\sin^2\theta + \cos^2\theta = 1$

Substitute: $4 \cdot \left(\frac{x^2}{4} + \frac{y^2}{4} = 1\right)$

Do not need to add domain restriction b/c the conic only graphs [-2,2]

$x^2 + y^2 = 4$ Circle w/r=2

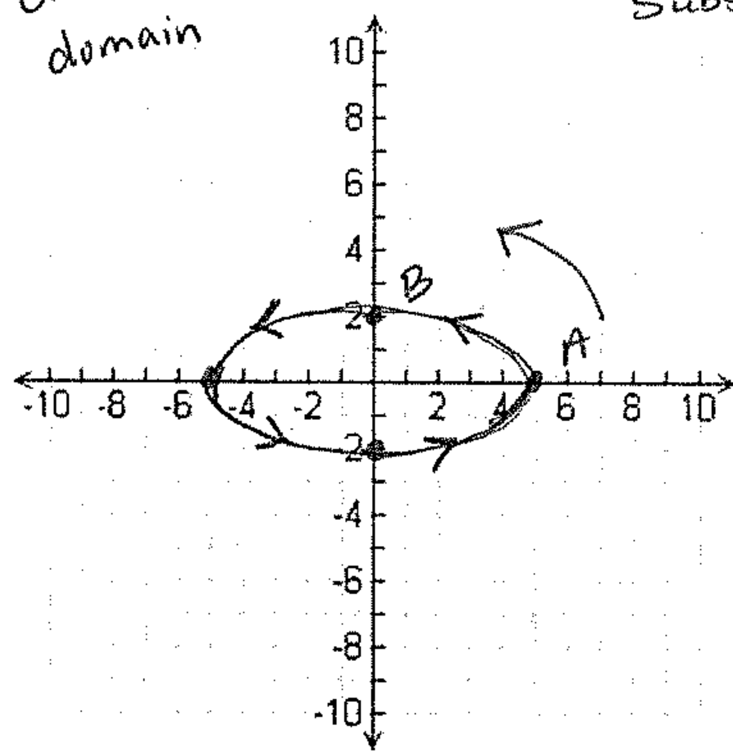
Ex.5: $x = 5\cos\theta \qquad y = 2\sin\theta$

θ domain all reals

$x \in [-5, 5]$ Restriction?

plane curve's domain

θ	x	y	
0	(5,	0)	A
$\frac{\pi}{2}$	(0,	2)	B
π	(−5,	0)	
$\frac{3\pi}{2}$	(0,	−2)	



Eliminate Parameter (θ)

$(x)^2 = (5\cos\theta)^2$

$x^2 = 25\cos^2\theta$

$\frac{x^2}{25} = \cos^2\theta$

$(y)^2 = (2\sin\theta)^2$

$y^2 = 4\sin^2\theta$

$\frac{y^2}{4} = \sin^2\theta$

Substitute:

$\cos^2\theta + \sin^2\theta = 1$

$$\boxed{\frac{x^2}{25} + \frac{y^2}{4} = 1}$$

$\boxed{\text{Ellipse}}$

no restriction needed for an ellipse

$x = 4\cos\theta$

$\cancel{[-4, 4]}$

Ex.6: $x = 4\sec t \qquad y = 3\tan t$

$x \in (-\infty, -4] \cup [4, \infty)$

Restriction?

(t) domain has restrictions b/c sect and tant are undefined at some angles.

t	x	y
0	$(4,$	$0)$ A
$\frac{\pi}{3}$	$(8,$	$3\sqrt{3})$ B
$\frac{\pi}{2}$	Und	Und
	.	
π	$(-4$	$0)$

need to add in →

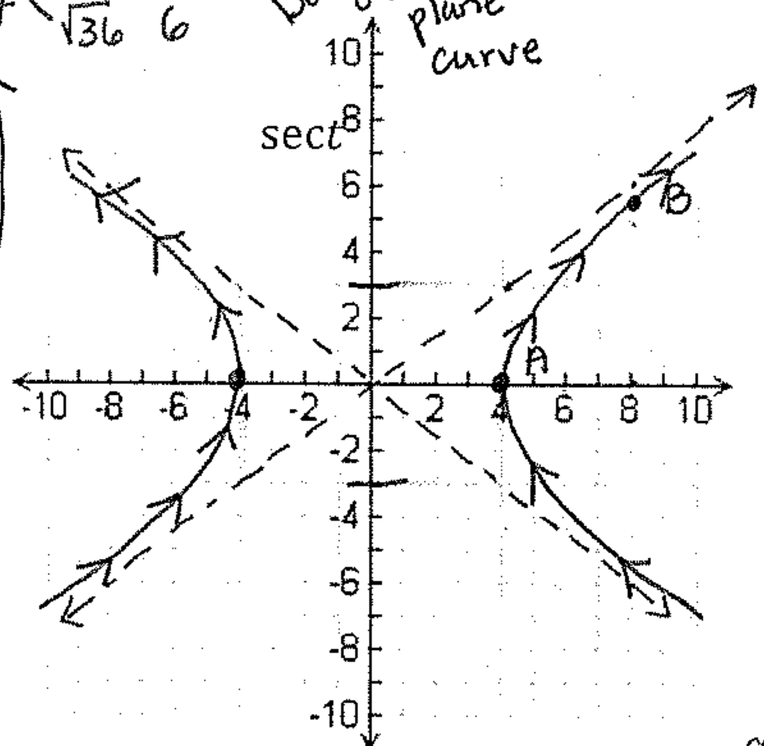
$y = 3\tan\frac{\pi}{3}$

$y = 3\sqrt{3}$

$y = \sqrt{9 \cdot 3}$

$y = \sqrt{27}$ — $\sqrt{25}$ 5, $\sqrt{36}$ 6

Domain of the plane curve



We will need to eliminate the parameter (t) to complete this graph.

$(x)^2 = (4\sec t)^2$

$x^2 = 16\sec^2 t$

$\frac{x^2}{16} = \sec^2 t$

$(y)^2 = (3\tan t)^2$

$y^2 = 9\tan^2 t$

$\frac{y^2}{9} = \tan^2 t$

• Substitute:

$\tan^2\theta + 1 = \sec^2\theta$

$\frac{y^2}{9} + 1 = \frac{x^2}{16}$

$1 = \frac{x^2}{16} - \frac{y^2}{9}$

$$\boxed{\frac{x^2}{16} - \frac{y^2}{9} = 1}$$

$\boxed{\text{Hyperbola}}$

$a = 4 \qquad b = 3$

Ex.7: $x = 4^{-t}$ $\quad y = 4^{2t}$

$x = (4^{-1})^t$

$x = \left(\frac{1}{4}\right)^t$

$y \in (0, \infty)$

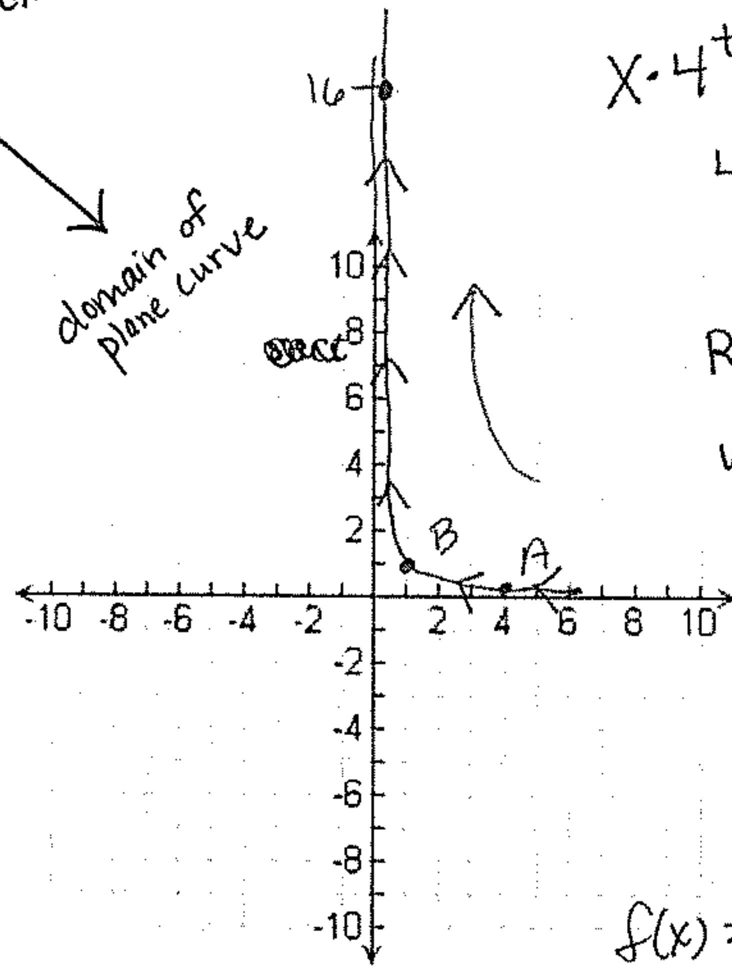
(t) domain all reals

range $x \in (0, \infty)$

Restriction?

domain of plane curve

t	x	y	
-1	$(4$	$\frac{1}{16})$	A
0	$(1,$	$1)$	B
1	$(\frac{1}{4},$	$16)$	



Eliminate the parameter (t):

$x = \left(\frac{1}{4}\right)^t$

$\frac{x}{1} = \frac{1}{4^t}$

$x \cdot 4^t = 1$

$4^t = \frac{1}{x}$

$y = 4^{2t}$

$y = (4^t)^2$

$y = \left(\frac{1}{x}\right)^2$

Reciprocal Squared w/ restriction

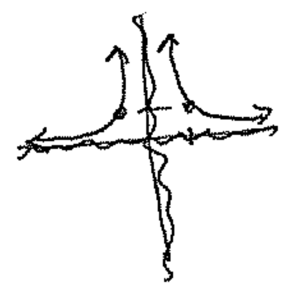
$$\boxed{y = \frac{1}{x^2};\ x > 0}$$

rectangular equation

must add on the domain restriction

$f(x) = \frac{1}{x^2}$

LOF



8) $x = 4\cos^{2}\theta$ $\qquad$ $y = 2\sin\theta$

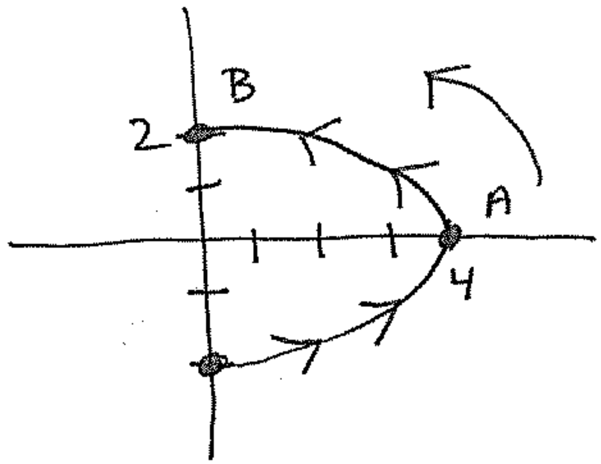
$x = 4(\cos\theta)^2$ $\qquad$ $y \in [-2, 2]$

θ domain all reals

$x \in [\cancel{-4}, 4]$

$x \in [0, 4]$

θ	x	y	
0	(4,	0)	A
$\frac{\pi}{2}$	(0,	2)	B
π	(4,	0)	
$\frac{3\pi}{2}$	(0,	−2)	



Eliminate the Parameter (θ):

$x = 4\cos^2\theta$ ← already squared

$\frac{x}{4} = \cos^2\theta$

$(y)^2 = (2\sin\theta)^2$

$y^2 = 4\sin^2\theta$

$\frac{y^2}{4} = \sin^2\theta$

$\boxed{\text{Parabola w/ restriction}}$

$\cos^2\theta + \sin^2\theta = 1$

$\frac{x}{4} + \frac{y^2}{4} = 1$

$x + y^2 = 4$

$\boxed{x = -y^2 + 4;\ 0 \le x \le 4}$

or

$y^2 = -x + 4$ $\quad$ $\boxed{y^2 = -(x-4);\ 0 \le x \le 4}$

$2t > 0$
$t > 0$

9) $x = \ln 2t \longrightarrow x = \log_e 2t \quad x \in (-\infty, \infty)$

$y = t^2$

Eliminate the parameter:

$x = \ln 2t$

$e^x = e^{\log_e 2t}$

$e^x = 2t$

$\frac{e^x}{2} = t$

$y = t^2$

$\downarrow$

$y = \left(\frac{e^x}{2}\right)^2$

$\boxed{y = \frac{e^{2x}}{4}}$ no restriction needed

or

$\boxed{y = \frac{1}{4}e^{2x}}$