Notes

Pre-Calculus
Sec. 5.3
Double Angle and Power Reducing Formulas

Double Angle Formulas

$$\sin 2x = 2\sin x \cos x$$

$\sin 2x \rightarrow \sin(\overset{\alpha}{x} + \overset{\beta}{x})$ expand w/ sine sandwich (sum of sine)

$= \sin \overset{\alpha}{x} \cos \overset{\beta}{x} + \cos \overset{\alpha}{x} \sin \overset{\beta}{x}$ reorder

$= 1\sin x \cos x + 1\sin x \cos x$

$\sin 2x = 2\sin x \cos x$

Double Angle Formulas

$$\cos 2x = \cos^2 x - \sin^2 x$$ #1

$\cos 2x \rightarrow \cos(\overset{\alpha}{x} + \overset{\beta}{x})$ expand with sum of cosine

$= \cos x \cos x - \sin x \sin x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$\cos 2x = 2\cos^2 x - 1$$

use the 1st one → $\cos^2 x - \sin^2 x$

w/ pythagorean substitution

$\cos^2 x - (1 - \cos^2 x)$

$\cos^2 x - 1 + \cos^2 x \longrightarrow 2\cos^2 x - 1$

$$\cos 2x = 1 - 2\sin^2 x$$

use the 1st one → $\cos^2 x - \sin^2 x$

w/ pythagorean substitution

$1 - \sin^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

Double Angle Formulas

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\tan 2x \rightarrow \tan(\overset{\alpha}{x} + \overset{\beta}{x})$$

expand
with sum
of tangent

$$= \frac{\tan x + \tan x}{1 - \underbrace{\tan x \tan x}}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

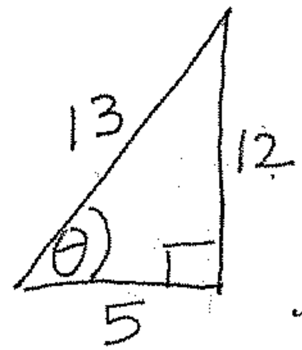
$$= 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Ex. 1: Given a right triangle with a side of 5, hypotenuse of 13, and θ the angle between them; find the following:

$a)\ \sin\theta = \boxed{\frac{12}{13}}$

SOH



triplet:

5-12-13

$b)\ \cos\theta = \boxed{\frac{5}{13}}$

CAH

$c)\ \sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{2}{1}\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$

$= \boxed{\frac{120}{169}}$

$d)\ \cos 2\theta$

$= \cos^2\theta - \sin^2\theta$

$= (\cos\theta)^2 - (\sin\theta)^2$

$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2$

$= \frac{25}{169} - \frac{144}{169} = \boxed{-\frac{119}{169}}$

$e)\ \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{120}{169}}{\frac{-119}{169}} = \boxed{-\frac{120}{119}}$

or use the tan2θ formula

$f)\ \cot 2\theta = \boxed{-\frac{119}{120}}$

the reciprocal of tan 2θ

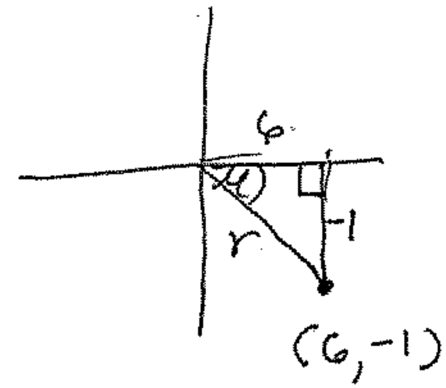
Ex. 2: Find the exact value of cos2u, using the double angle formulas.

$$Given: \quad \cot u = -6 \quad and \quad \frac{3\pi}{2} < u < 2\pi$$

QIV

$\tan u = -\frac{1}{6}$

TOA



$$\cos 2u = \cos^2 u - \sin^2 u$$
$$= (\cos u)^2 - (\sin u)^2$$
$$= \left(\frac{6}{\sqrt{37}}\right)^2 - \left(\frac{-1}{\sqrt{37}}\right)^2$$
$$= \frac{36}{37} - \frac{1}{37}$$
$$= \boxed{\frac{35}{37}}$$

$36 + 1 = r^2$

$37 = r^2$

$\sqrt{37} = r$

$\cos u = \frac{6}{\sqrt{37}}$

CAH

QIV

$\sin u = \frac{-1}{\sqrt{37}}$

SOH

Power Reduction Formulas

These new formulas are derived from the Cosine Double Angle Formulas

Recall:

$$\cos \underline{\underline{2u}} = \cos^2 u - \sin^2 u$$

$$= 2\cos^2 u - 1$$

$$= 1 - 2\sin^2 u$$

The Power Reducing Formulas:

so…

$$2\cos^2 u - 1 = \cos 2u$$

← cosine double angle formula

$$2\cos^2 u = 1 + \cos 2u$$

$$\boxed{\cos^2 u = \frac{1 + \cos 2u}{2}}$$

Reduce the power, and double the angle

The Power Reducing Formulas:

so…

$$1-2\sin^2 u = \cos 2u$$ ← cosine double angle formula

multiply by (−1) → $$-2\sin^2 u = -1+\cos 2u$$

$$2\sin^2 u = 1-\cos 2u$$

$$\boxed{\sin^2 u = \frac{1-\cos 2u}{2}}$$

Reduce the power, and double the angle.

The Power Reducing Formulas:

so…

power reduction formulas of sine/cosine

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{\frac{1-\cos 2u}{\cancel{2}}}{\frac{1+\cos 2u}{\cancel{2}}} = \frac{1-\cos 2u}{1+\cos 2u}$$

$$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

Reduce the power, double the angle

Ex.1:Rewrite the expression in terms of the 1st power of cosine.

a) $\cos^2 \underbrace{(2x)}_{u} = \boxed{\begin{matrix}\text{Rule} \\ \dfrac{1+\cos 2u}{2}\end{matrix}} = \dfrac{1+\cos(2\cdot 2x)}{2}$

$= \boxed{\dfrac{1+\cos 4x}{2}}$ or $\boxed{\frac{1}{2}(1+\cos 4x)}$

b) $\sin^2 \underbrace{\left(\frac{x}{2}\right)}_{u} = \boxed{\begin{matrix}\text{Rule} \\ \dfrac{1-\cos 2u}{2}\end{matrix}} = \dfrac{1-\cos(\cancel{2}\cdot \frac{x}{\cancel{2}})}{2}$

$= \boxed{\dfrac{1-\cos x}{2}}$ or $\boxed{\frac{1}{2}(1-\cos x)}$

$\frac{x}{2} \Rightarrow \frac{1}{2}x$

c) $\sin^4 x = \sin^2 x \cdot \sin^2 x$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2x)(1-\cos 2x)$$

FOIL

Power reduces again!

angle $2x \to 4x$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)$$

☺

$$= \frac{1}{4}\left(\frac{2}{2}\cdot\frac{1}{1} - \frac{2}{2}\cdot\frac{2\cos 2x}{1} + \frac{1+\cos 4x}{2}\right)$$

get a common denominator

$$= \frac{1}{4}\left(\frac{2}{2} - \frac{4\cos 2x}{2} + \frac{1}{2} + \frac{\cos 4x}{2}\right)$$

$$= \frac{1}{4}\cdot\frac{1}{2}\left(2 - 4\cos 2x + 1 + \cos 4x\right)$$

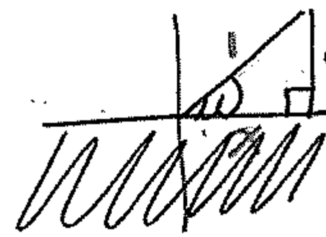
$$= \boxed{\frac{1}{8}\left(3 - 4\cos 2x + \cos 4x\right)} \text{ or } \boxed{\frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}}$$

Ex.2: Write the trigonometric expression as an algebraic expression.

angle

$\sin(2\arccos x)$

Let $u = \text{arc}\cos x$



$a = \sqrt{1-x^2}$

so...

$\sin 2u$ (double angle)

$= 2\sin u\cos u$

$= 2(\sqrt{1-x^2})(x)$

$= \boxed{2x\sqrt{1-x^2}}$

$\cos u = \frac{x}{1}$ $\frac{\text{adj}}{\text{hyp}}$.

CAH

$a^2 + x^2 = (1)^2$

$a^2 + x^2 = 1$

$a^2 = 1 - x^2$

$a = \oplus\sqrt{1-x^2}$

$\sin u = \frac{\sqrt{1-x^2}}{1}$ $\frac{\text{opp}}{\text{hyp}}$

SOH

Recall:

$\sin 2u$

$= \sin(u+u)$

$= \sin u\cos u + \cos u\sin u$

reorder

$= 2\sin u\cos u$

$\sqrt{1 \ominus x^2}$

$\sqrt{(1+x)(1-x)}$

Ex. 3: Verify the Identity:

a) $$\sec 2\theta = \frac{\sec^2\theta}{2-\sec^2\theta}$$

which one? →

$$\frac{1}{\cos 2\theta} = \frac{\frac{1}{\cancel{\cos^2\theta}}\cdot \cancel{\cos^2\theta}}{\cos^2\theta\cdot\frac{2}{1} - \frac{1}{\cancel{\cos^2\theta}}\cdot\cancel{\cos^2\theta}}$$

$$\frac{1}{\cos^2\theta - \sin^2\theta} = \boxed{\frac{1}{2\cos^2\theta - 1}}$$

pyth

$$\frac{1}{\cos^2\theta - (1-\cos^2\theta)} =$$

$$\frac{1}{\cos^2\theta - 1 + \cos^2\theta} =$$

$$\boxed{\frac{1}{2\cos^2\theta - 1}} =$$

✓

★ faster if you pick the one only in terms of $\cos\theta$

b) $1+\cos \underline{\underline{10x}} = 2\cos^2 \underline{\underline{5x}}$ can use Power reduction

$$= \cancel{2}\left(\frac{1+\cos 10x}{\cancel{2}}\right)$$

$$= 1+\cos 10x$$

c) $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos(\overset{\alpha}{2x} + \overset{\beta}{x}) =$$

$$\cos 2x \cos x \overset{\text{opp}}{-} \sin 2x \sin x =$$

$$(2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x =$$

$$2\cos^3 x - \cos x - 2\underset{\text{pyth}}{\sin^2 x}\cos x =$$

$$2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) =$$

$$2\cos^3 x - 1\cos x - 2\cos x + 2\cos^3 x =$$

$$\boxed{4\cos^3 x - 3\cos x} =$$