Notes

Pre-Calculus
Sec. 1.8
Inverse Functions

Definition of the Inverse Function

Given $f(x)$ and $g(x)$:

SHOW OR PROVE OR DECIDE INVERSES: NOT BY SWITCHING X & Y !!!

If $f(g(\mathbf{x})) = \mathbf{x}$ and $g(f(\mathbf{x})) = \mathbf{x}$,

$f \circ g(x) =$ $g \circ f(x) =$

then the function g is the **inverse of the function** f.

inverse function

We use $f^{-1}(x)$ as inverse notation.

The **domain** of f is equal to the **range** of f^{-1}, and vice versa.

(BECAUSE X & Y are switched.)

A function that has an inverse function is also called a one-to-one Function (passes both the VLT and the HLT).

vertical line test

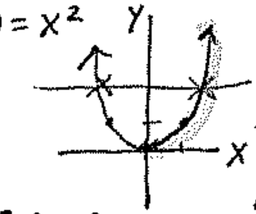
horizontal line test

To find the inverse of a set of points: switch x with y.
ie. {(1, 2), (3, 4)} $\longrightarrow$ {(2, 1), (4, 3)}
$f(x)$ $f^{-1}(x)$

X's do not repeat

One-to-One Function: a *f* (which already passes the vertical line test) must also pass the horizontal line test (HLT) (Now Y's do not repeat either).

$f(x) = x^2$



Passes VLT
Fails HLT
$f(x)$ is not a one-to-one function, the inverse is NOT a function.

* If a domain restriction is added ie. $x \geq 0$ $f(x)$ will be one-to-one.

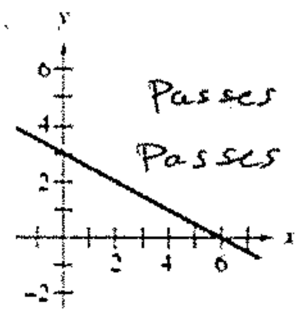
A function *f* has an inverse (f^{-1}), iff it is one-to-one (passes both the VLT and the HLT) (if and only if)

To "**find**" the inverse of an equation, that is a one-to-one function, you will switch *x* & *y*, then solve for *y*.

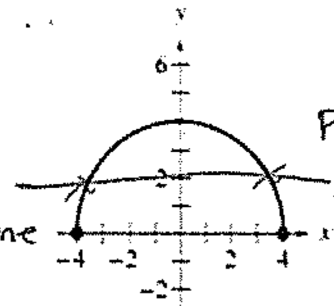
Ex.1

Horizontal Line Test and One-to-One Functions

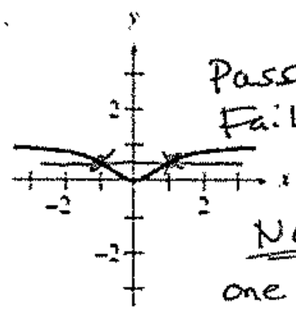
Part 1 – Does this function have an inverse function? (I.E. is it One-to-One?)



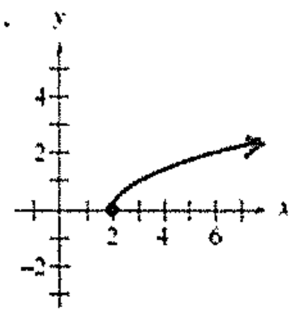
Passes VLT
Passes HLT
one-to-one
Yes



Passes VLT
Fails HLT
NOT one-to-one
NO



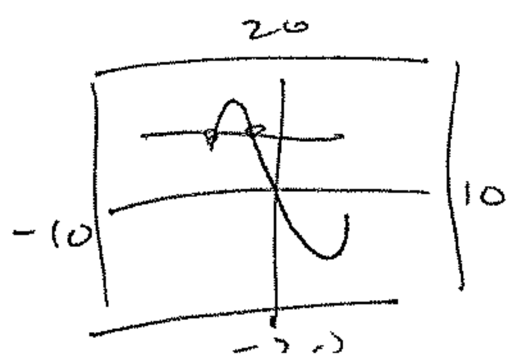
Passes VLT
Fails HLT
NOT
one-to-one
NO



Passes VLT + HLT
one-to-one
Yes

Part 2 – Use a graphing calculator to graph the function and determine if it is One-To-One (I.E. it has an inverse function)

$h(x) = -2x\sqrt{16-x^2}$



h(x)
Passes VLT and
Fails HLT
∴ NOT one-to-one

Ex.2

$f \circ g(x) = x$ and $g \circ f(x) = x$

[redacted] (a) show that f and g are inverse functions algebraically and (b) [redacted] (by hand) create a table of values for each function to numerically show that f and g are inverse functions.

[redacted] $f(x) = \dfrac{x - 9}{4}$, $g(x) = 4x + 9$

(a) $f \circ g(x)$

$= f(g(x))$

$= f(4x+9)$

$= \dfrac{(4x+\cancel{9})-\cancel{9}}{4}$

$= \dfrac{\cancel{4}x}{\cancel{4}}$

$= x$

$g \circ f(x)$

$= g(f(x))$

$= g\left(\dfrac{x-9}{4}\right)$

$= \cancel{4}\left(\dfrac{x-9}{\cancel{4}}\right)+9$

$= x - \cancel{9} + \cancel{9}$

$= x$

$\boxed{f \circ g(x) = x = g \circ f(x)}$

∴ $f(x)$ and $g(x)$ are inverses

(b) numerically (table)

x	9	13	10
$f(x)$	0	1	$\frac{1}{4}$

x	0	1	$\frac{1}{4}$
$g(x)$	9	13	10

∴ $f(x)$ & $g(x)$ are inverses.

Ex.3

~~In Exercises 15–20~~, show that f and g are inverse functions algebraically. ~~Use a graphing utility to~~ By hand graph f and g in the same viewing window. Describe the relationship between the graphs.

$f(x) = 9 - x^2$, $x \geq 0$; $g(x) = \sqrt{9 - x}$

ⓐ

$f \circ g(x)$	$g \circ f(x)$
$f(g(x))$	$g(f(x))$
$= f(\sqrt{9-x})$	$= g(9-x^2)$
$= 9-(\sqrt{9-x})^2$	$= \sqrt{9-(9-x^2)}$
* $= 9-(9-x)$	$= \sqrt{\cancel{9}-\cancel{9}+x^2}$
$= \cancel{9}-\cancel{9}+x$	$= \sqrt{x^2}$
$= x$	$= \lvert x \rvert$
	$= \pm x$
	$= x$ b/c $x \geq 0$

• look at domain of result and inside function

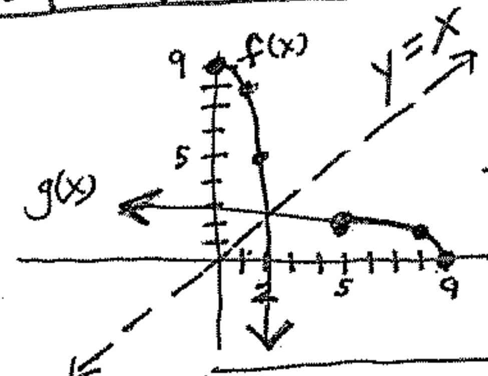
$\boxed{f \circ g(x) = x = g \circ f(x) \therefore f(x) \text{ \& } g(x) \text{ are inverses.}}$

ⓑ Graph

$f(x) = 9 - x^2$
$= -x^2 + 9$; $x \geq 0$

x	0	1	2
$f(x)$	9	8	5

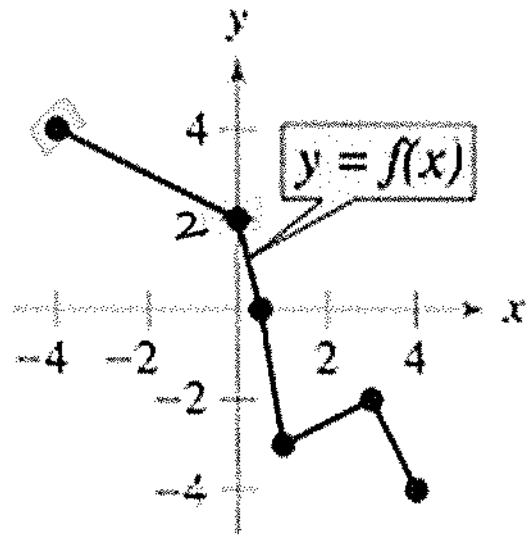
x	9	8	5
$g(x)$	0	1	2

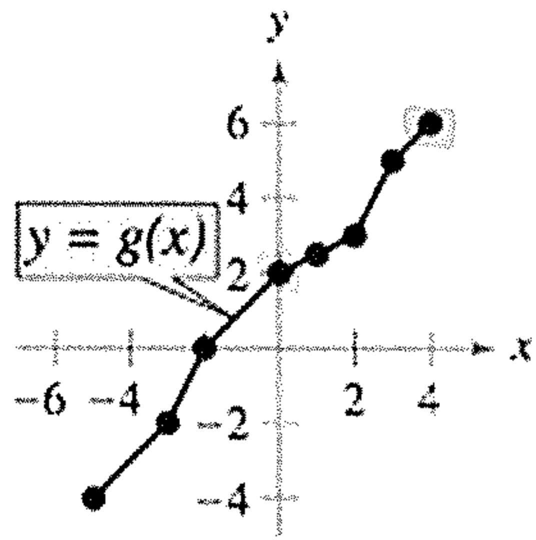


f(x) & g(x) are reflections over the line y=x ∴ f(x) & g(x) are inverses.

Ex.4

[redacted] use the graphs of $y = f(x)$ and $y = g(x)$ to evaluate the function.





84. $g(\underline{f(-4)})$

$= g(\underset{x}{4})$

$= \boxed{6}$

Find $f(\underset{x}{-4}) = 4$

87. $(g \circ f^{-1})(2)$

$= g(\underline{f^{-1}(2)})$

$= g(\underset{x}{0}) = \boxed{2}$

Find $f^{-1}(2)$

$(2, \underline{0})$

so . . .

on $f(x)$

$(\underline{0}, 2)$

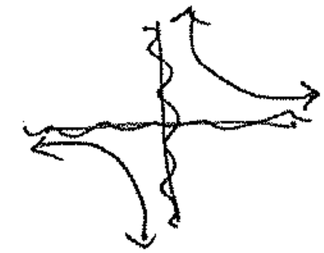
$\Uparrow$ y-value

Ex.5 a) Find the inverse of: $f(x) = \dfrac{3}{2+5x}$

y

- Switch x & y
- Then solve for y.

(LOF) $y = \frac{1}{x}$



switch: $\frac{x}{1} = \frac{3}{2+5y}$

distribute: $x(2+5y) = 3$

$2x + 5xy = 3$

$\frac{\cancel{5x}y}{\cancel{5x}} = \frac{3-2x}{5x}$

$y = \frac{3-2x}{5x}$ no complex fractions!

Use inverse function notation

$$\boxed{f^{-1}(x) = \frac{3-2x}{5x}}$$

b) Give the domain and range of $f(x)$ *and* $f^{-1}(x)$ in interval notation.

$f(x) = \frac{3}{2+5x}$

HA: $y=0$

$2+5x=0$

$5x=-2$

$x \neq -\frac{2}{5}$

$f(x)$: D: $(-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, \infty)$

R: $(-\infty, 0) \cup (0, \infty)$

$f^{-1}(x) = \frac{3-2x}{5x}$

domain: $5x=0$

$x \neq 0$

HA: $y = -\frac{2}{5}$

$f^{-1}(x)$: D: $(-\infty, 0) \cup (0, \infty)$

R: $(-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, \infty)$

Ex.6 a) Find $f^{-1}(x)$ for

$$\underset{y}{f(x)} = \frac{2x+3}{x-1}$$

$$\frac{x}{1} \cancel{=} \frac{2y+3}{y-1}$$

$$x(y-1) = 2y+3$$

$$xy - x = 2y + 3$$

$$xy - 2y = x + 3$$

Factor $\quad y(x-2) = x+3$

$$y = \frac{x+3}{x-2}$$

$$\boxed{f^{-1}(x) = \frac{x+3}{x-2}}$$

$$y = \frac{-x-3}{-x+2} = \frac{\cancel{-}(x+3)}{\cancel{-}(x-2)}$$

b) Give the domain and range of $f(x)$ and $f^{-1}(x)$ in interval notation.

$f(x) = \dfrac{2x+3}{x-1} \leftarrow x \neq 1$

BOBO
BOTN
EATS DC $\rightarrow$ HA: $y=2$

$f(x)$
D: $(-\infty, 1) \cup (1, \infty)$
R: $(-\infty, 2) \cup (2, \infty)$

$f^{-1}(x) = \dfrac{x+3}{x-2} \leftarrow x \neq 2$

HA: $y=1$

$f^{-1}(x)$
D: $(-\infty, 2) \cup (2, \infty)$
R: $(-\infty, 1) \cup (1, \infty)$

Ex.7 a) Find the inverse of $f(x) = \sqrt{5-x}$

$y = \sqrt{5-x}$

$(x)^2 = (\sqrt{5-y})^2$

$x^2 = 5 - y$

$x^2 - 5 = -y$

$-x^2 + 5 = y$

$$\boxed{f^{-1}(x) = -x^2 + 5 \ ; \ x \geq 0}$$

must add a restriction to be one-to-one

b) Give the domain and range of $f(x)$ and $f^{-1}(x)$ in interval notation.

$f(x) = \sqrt{5-x}$

$5 - x \geq 0$

$-x \geq -5$

$x \leq 5$

$f(x)$:

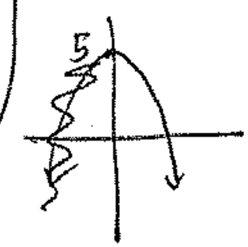
D: $(-\infty, 5]$

R: $[0, \infty)$

$f^{-1}(x)$

D: $[0, \infty)$

R: $(-\infty, 5]$



Ex. 8 Given:

$$f(x)=\frac{1}{8}x-3, \quad g(x)=x^3, \quad h(x)=2x+1$$

a) Find $(f \circ g)^{-1}(5)$

Need $(f \circ g)(x)$ to find $(f \circ g)^{-1}(x)$

$$(f \circ g)(x) = f(g(x))$$
$$= f(x^3)$$
$$= \frac{1}{8}(x^3)-3$$

$$\underbrace{(f \circ g)(x)}_{Y} = \frac{x^3}{8}-3$$

Switch x & Y to find $(f \circ g)^{-1}(x)$

$$x = \frac{Y^3}{8}-3$$

$$8(x+3) = \frac{Y^3}{\cancel{8}} \cdot \cancel{8}$$

$$\sqrt[3]{Y^3} = \sqrt[3]{8\cdot(x+3)}$$

$$Y = 2\sqrt[3]{x+3}$$

$$(f \circ g)^{-1}(x) = 2\sqrt[3]{x+3}$$

All reals

Inside: $g(x)=x^3$ All reals

$$(f \circ g)^{-1}(5) = 2\sqrt[3]{5+3}$$
$$= 2\sqrt[3]{8}$$
$$= 2 \cdot 2$$
$$= \boxed{4}$$

$$f(x) = \frac{1}{8}x - 3, \quad g(x) = x^3, \quad h(x) = 2x + 1$$

b) Find $h(g(f(8)))$

$= h(g(-2))$

$= h(-8)$

$= 2(-8) + 1$

$= -16 + 1$

$= \boxed{-15}$

1st find: $f(8) = \frac{1}{\cancel{8}}(\cancel{8}) - 3$

$= \boxed{-2}$

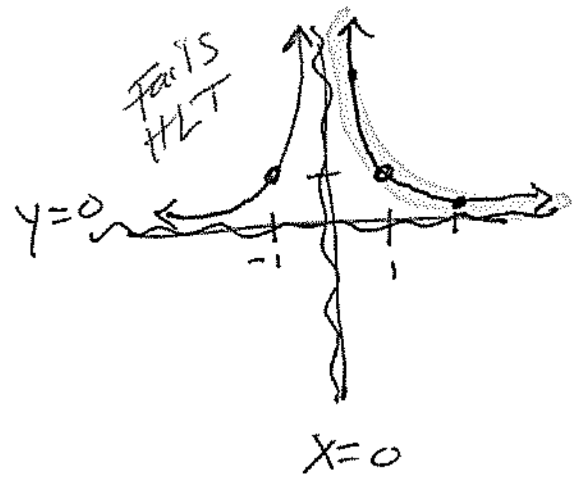
Now find $g(-2) = (-2)^3 = \boxed{-8}$

Ex 9) Find the inverse of $f(x) = \frac{1}{x^2}$; $x > 0$

y

$x \neq 0$

even power



HA: $y=0$

$$\frac{x}{1} = \frac{1}{y^2}$$

$$xy^2 = 1$$

$$\sqrt{y^2} = \pm\sqrt{\frac{1}{x}}$$

$$y = \pm \frac{1}{\sqrt{x}}$$

not one-to-one

$f(x)$:

D: $(0, \infty)$

R: $(0, \infty)$

$$\boxed{f^{-1}(x) = \frac{1}{\sqrt{x}}, \ x > 0}$$

add a restriction

$f^{-1}(x)$:

D: $(0, \infty)$

R: $(0, \infty)$