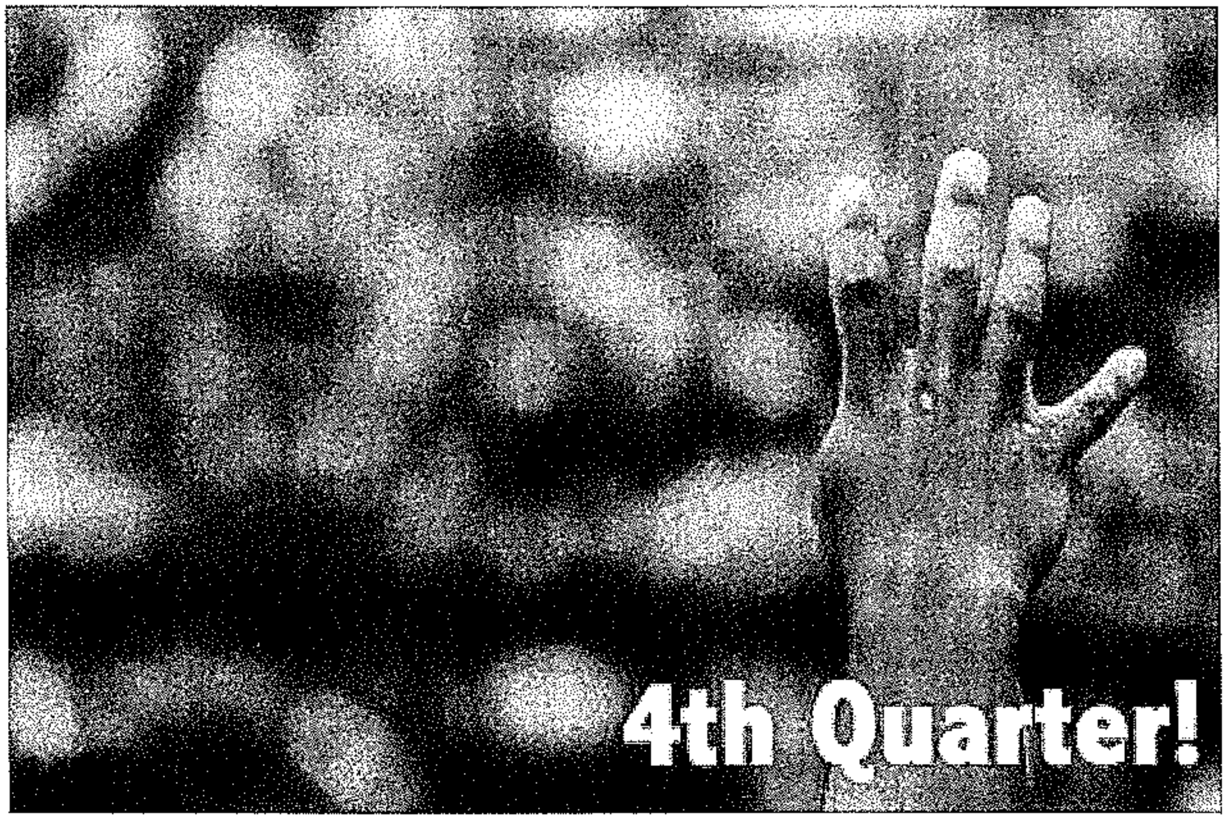
Inverse Functions

Notes



Inverse Functions - **Algebraic** Definition

f and g are inverse functions if

$$(f \circ g)(x) = x$$

AND

$$(g \circ f)(x) = x$$

Proving/showing

Verifying Inverse Functions

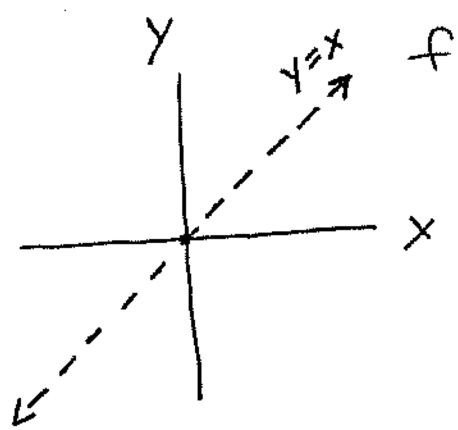
1. **Algebraically**

Show: $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$

Verifying Inverse Functions

2. **Graphically**

Show: There is a reflection over the line $y=x$, by graphing both $f(x)$ and $g(x)$ on the same grid.



Verifying Inverse Functions

3. **Numerically** (NOT A PROOF)

Show:

1st function $f(x)$

x	y
−2	0
1	3
4	6

plug into

2nd function $g(x)$

x	y
0	−2
3	1
6	4

If the x & y switched; you have inverses.

verify/prove

ex: Show $f(x) = 4x + 9$ and $g(x) = \dfrac{x-9}{4}$ are inverses, algebraically.

$f \circ g(x)$	$g \circ f(x)$
$f(g(x)) = f\left(\frac{x-9}{4}\right)$	$g(f(x)) = g(4x+9)$
$\Rightarrow 4x + 9$	$\Rightarrow \frac{x-9}{4}$
$= \cancel{4}\left(\frac{x-9}{\cancel{4}}\right) + 9$	$= \frac{(4x+\cancel{9}) - \cancel{9}}{4}$
$= x - \cancel{9} + \cancel{9}$	$= \frac{\cancel{4}x}{\cancel{4}}$

$f(g(x)) = x$ ✓ and $g(f(x)) = x$ ✓

so... $\boxed{f(x) \text{ and } g(x) \text{ are inverses.}}$

ex: If f and g are inverse functions and f contains the point $(2, -3)$ then g must contain the point $\underline{\quad(-3, 2)\quad}$.

ex: Are f and g inverses?

$$f(x) = x^2 \text{ and } g(x) = \sqrt{x}$$

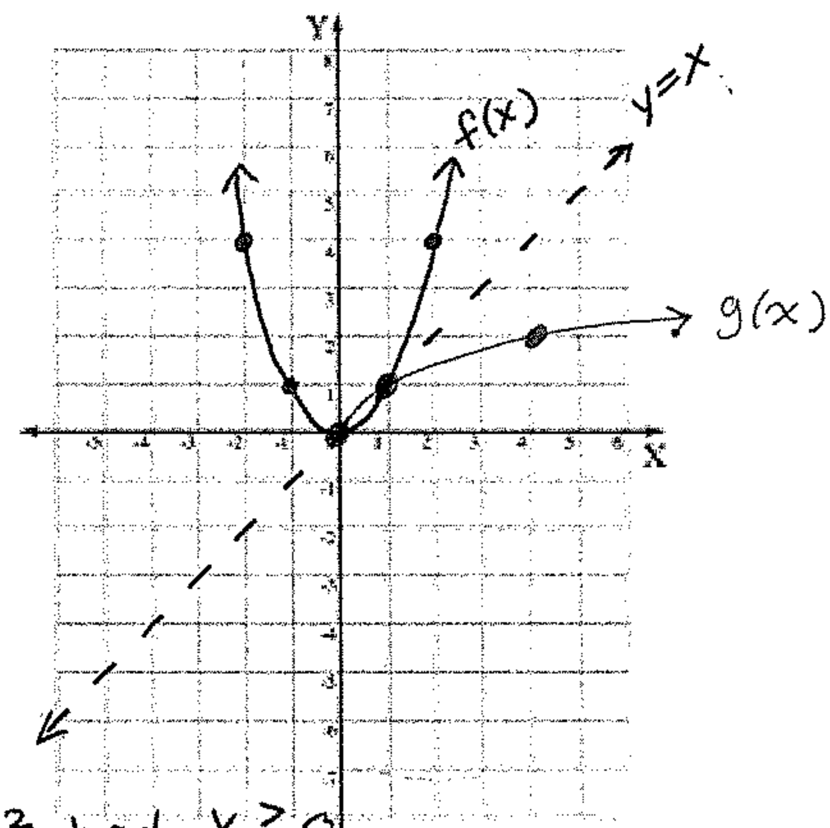
a) Prove your answer graphically.

$f(x) = x^2$
parabola

x	y
-2	4
-1	1
0	0
1	1
2	4

$g(x) = \sqrt{x}$

x	y
* 0	0
1	1
4	2
9	3



$\boxed{\text{f(x) and g(x) are NOT inverses.}}$

But... if $f(x) = x^2$ had $x \geq 0$
then they would be inverses.

ex: Are f and g inverses?

$$f(x) = x^2 \text{ and } g(x) = \sqrt{x}$$

b) Prove your answer algebraically.

$f \circ g(x)$	$g \circ f(x)$
$f(g(x)) = f(\sqrt{x})$	$g(f(x)) = g(x^2)$
$\Rightarrow x^2$	$\Rightarrow \sqrt{x}$
$= (\sqrt{x})^2$	$= \sqrt{x^2}$ (even root)

$f(g(x)) = x$ ✓ and $g(f(x)) = |x|$ ✗ No

* unless a domain restriction is added

so... $\boxed{f(x) \text{ and } g(x) \text{ are NOT inverses}}$

ex: Are f and g inverses?

$$f(x) = 2x - 5 \qquad g(x) = \frac{x+5}{2}$$

a) Prove your answer graphically.

$g(x) = \frac{x}{2} + \frac{5}{2}$

$m = \frac{1}{2}$

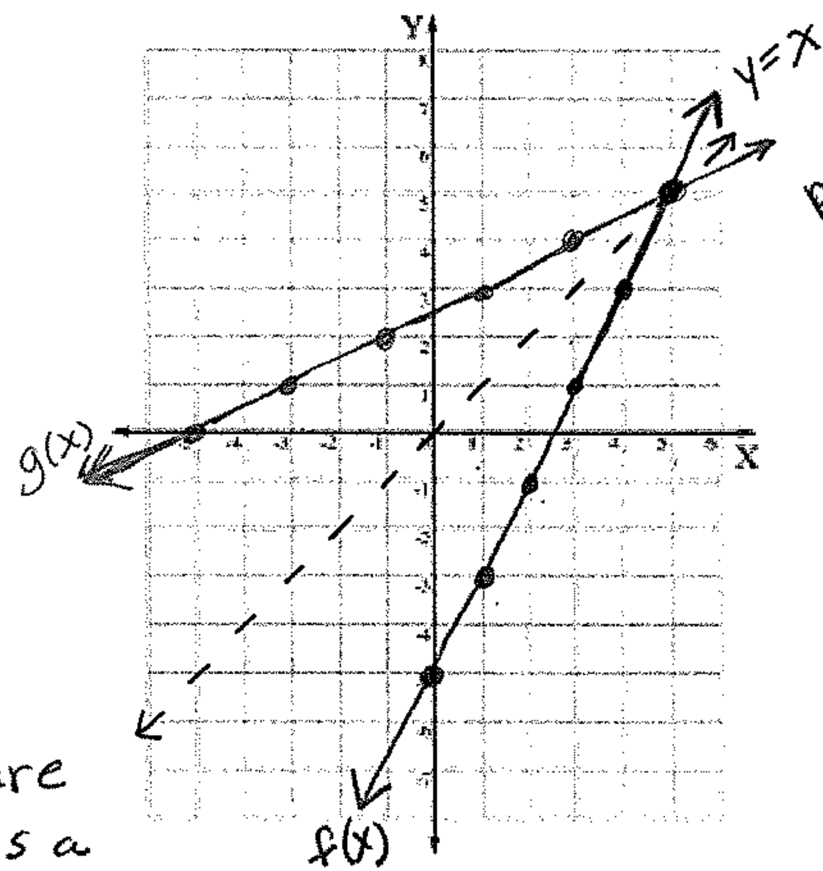
$f(x) = 2x - 5$

Lineer

y-int

$m = \frac{2}{1}$

x	y
0	-5
1	-3
2	-1
3	1
4	3
5	5



plug in the y's from f(x)

g(x)

x	y
-5	0
-3	1
-1	2
1	3
3	4
5	5

f(x) and g(x) are inverses. There is a reflection over the line y=x.

ex: Are f and g inverses?

$$f(x) = 2x - 5 \qquad g(x) = \frac{x+5}{2}$$

b) Prove your answer algebraically.

$f \circ g(x)$	$g \circ f(x)$
$f(g(x)) = f\left(\frac{x+5}{2}\right)$	$g(f(x)) = g(2x-5)$
$\Rightarrow 2x - 5$	$\Rightarrow \frac{x+5}{2}$
$= \cancel{2}\left(\frac{x+5}{\cancel{2}}\right) - 5$	$= \frac{(2x-\cancel{5}) + \cancel{5}}{2}$
$= x + \cancel{5} - \cancel{5}$	$= \frac{\cancel{2}x}{\cancel{2}}$

$f(g(x)) = x$ ✓ and $g(f(x)) = x$ ✓

so...

$\boxed{\text{f(x) and g(x) are inverses}}$

<u>The Existence of an Inverse</u>

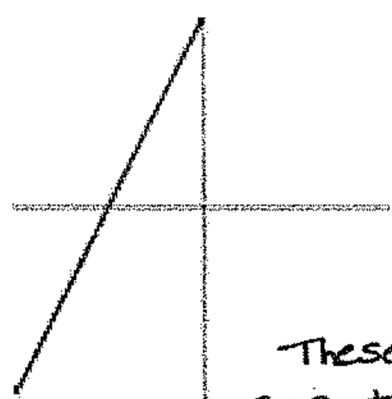
A function has an inverse function if it passes BOTH the vertical line test (VLT) and horizontal line test (HLT).

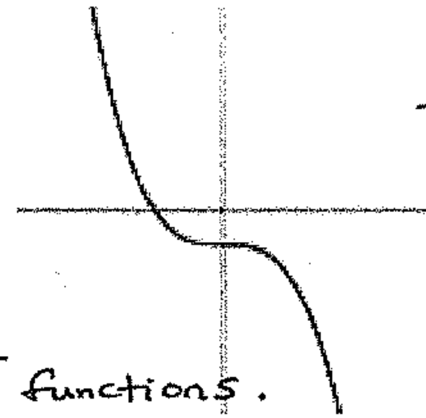
<u>One-To-One</u>

A function is <u>one-to-one</u> if it passes BOTH the vertical line test (VLT) and horizontal line test(HLT).

The Existence of an Inverse

Examples of functions that DO have inverses:

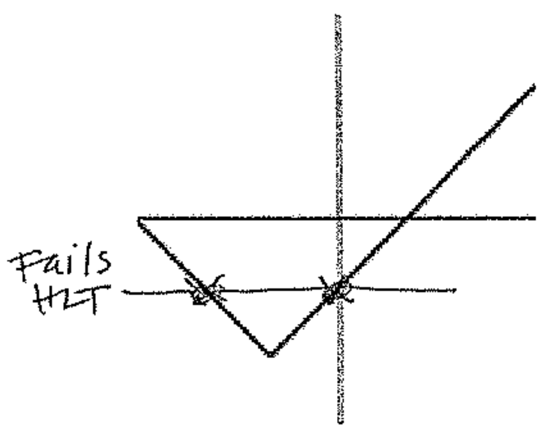


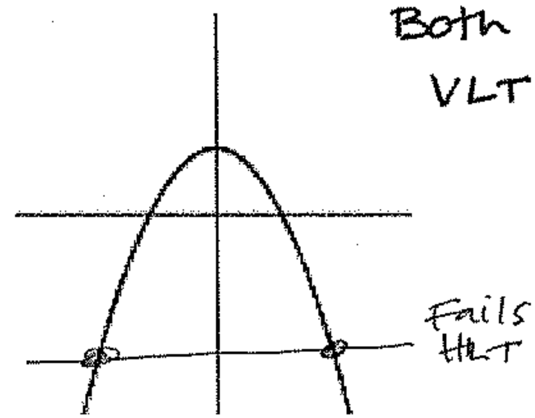


These are one-to-one functions.

Pass both the VLT (FUNCTIONS) and the HLT (the inverses are functions)

Examples of functions that DO NOT have inverse functions.





Both pass the VLT (Functions) But both FAIL the HLT (the inverses are NOT functions)

NOT one-to-one functions.

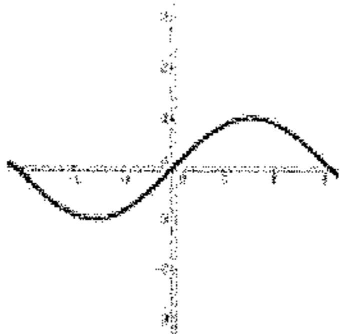
Inverse Notation

inverse

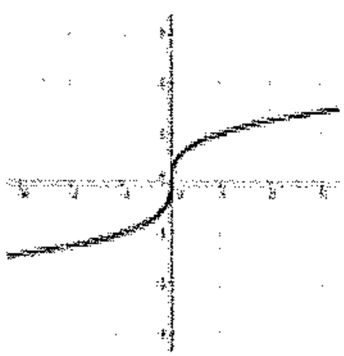
$$\boxed{f^{-1}(x)}$$

NOTE: $f^{-1}(x) \neq \left[f(x)\right]^{-1}$ AND $f^{-1}(x) \neq \dfrac{1}{f(x)}$

ex: Determine whether each function has an inverse function.

a) 

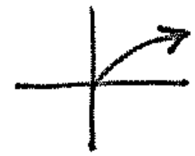
FAILS HLT,
Does **NOT** have
an inverse function.

b) 

Passes both VLT and HLT;
has an inverse function.

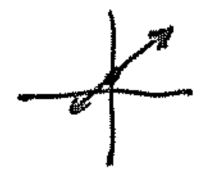
ex: Determine whether each function has an inverse function.

c) $f(x) = \sqrt{x}$



yes, passes VLT & HLT.

d) $f(x) = x + 1$ Linear



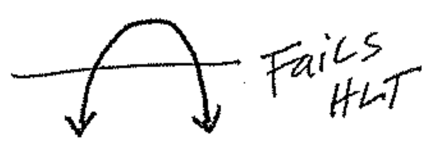
$y = mx + b$

yes, passes VLT & HLT.

e) $f(x) = -3x^2 + 4x + 5$

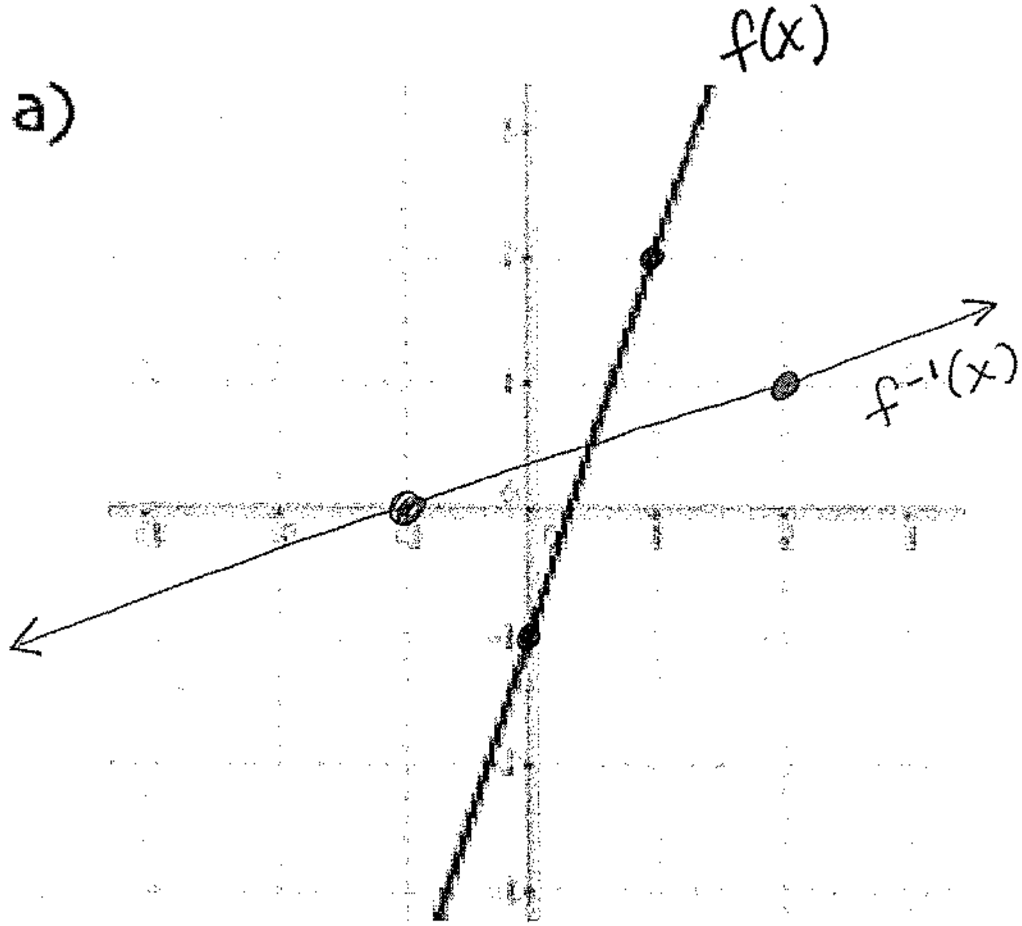
negative Parabola

Fails HLT, no inverse function.



ex: Sketch the inverse function, if it exists.

a)



✓ passes VLT & HLT

$f(x)$:

x	y
0	−1
1	2

switch x & y

$f^{-1}(x)$: ← inverse

x	y
−1	0
2	1

ex: Find the inverse, if possible.

a) $f(x)=\dfrac{5-3x}{2}$

Linear, has an inverse function.

$y = \frac{5}{2} - \frac{3x}{2}$

$y = -\frac{3}{2}x + \frac{5}{2}$

Linear

$y = \dfrac{5-3x}{2}$

switch x & y

$x = \dfrac{5-3y}{2}$

Solve for y

$2(x) = \left(\dfrac{5-3y}{\cancel{2}}\right)\cdot\cancel{2}$

$2x = \cancel{5} - 3y$

$-5 \quad \cancel{-5}$

$\dfrac{2x-5}{-3} = \dfrac{-\cancel{3}y}{\cancel{-3}}$

Linear:

$y = -\frac{2}{3}x + \frac{5}{3}$

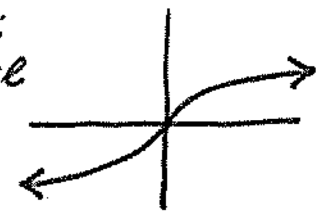
change to inverse notation

$$\boxed{f^{-1}(x) = -\frac{2}{3}x + \frac{5}{3}}$$

ex: Find the inverse, if possible.

b) $g(x) = \sqrt[3]{x+8}$

Cube roots have inverse functions.



$y = \sqrt[3]{x+8}$

switch x & y

$x = \sqrt[3]{y+8}$

Solve for y

$(x)^3 = \left(\sqrt[3]{y+8}\right)^3$

$x^3 = y + \cancel{8}$

$-8 \qquad \cancel{-8}$

$x^3 - 8 = y$

$y = x^3 - 8$

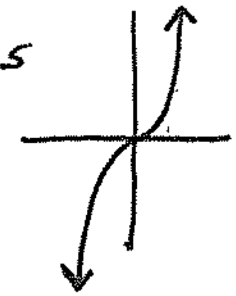
change to inverse notation

$\boxed{g^{-1}(x) = x^3 - 8}$

ex: Find the inverse, if possible.

c) $h(x) = 8x^3 + 7$

Cubic functions
$y = x^3$
pass VLT &
HLT
and have
inverse functions.



$$\downarrow$$

$$y = 8x^3 + 7$$

switch x & y

$$x = 8y^3 + 7$$

Solve for y

$$x = 8y^3 + \cancel{7}$$
$$-7 \qquad\qquad \cancel{-7}$$

$$\frac{x-7}{8} = \frac{\cancel{8}y^3}{\cancel{8}}$$

$$\frac{x-7}{8} = y^3$$

$$y^3 = \frac{x-7}{8}$$

$$\sqrt[3]{y^3} = \sqrt[3]{\frac{x-7}{8}}$$

$$y = \frac{\sqrt[3]{x-7}}{\sqrt[3]{8}}$$

change to inverse notation

$$y = \frac{\sqrt[3]{x-7}}{2}$$

$$\boxed{h^{-1}(x) = \frac{\sqrt[3]{x-7}}{2}}$$